KRYLOV-SUBSPACE BASED MODEL REDUCTION
OF NONLINEAR CIRCUIT MODELS
USING BILINEAR AND QUADRATIC-LINEAR
APPROXIMATIONS

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16th European Conference on Mathematics for Industry (ECMI)
Wuppertal, 28 July 2010

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Outline

1. Nonlinear Model Reduction

2. Model Reduction via Bilinear Approximations
   - Bilinear Control Systems
   - Krylov-Based Model Reduction
   - Interpolatory Model Reduction
   - Numerical Examples

3. Model Reduction via Quadratic-Bilinearizations
   - Quadratic-Bilinear Control Systems
   - Reduction Techniques
   - Numerical Examples

4. Outlook
Outline

1. Nonlinear Model Reduction
   - Model Reduction via Bilinear Approximations
     - Bilinear Control Systems
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2. Outlook

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Consider a large-scale state-nonlinear control system of the form

\[
\Sigma : \begin{cases}
\dot{x}(t) = f(x(t)) + Bu(t), \\
y(t) = Cx(t), \\
x(0) = x_0,
\end{cases}
\]

with \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) nonlinear, \( B \in \mathbb{R}^{n \times m}, \ C \in \mathbb{R}^{p \times n}, \ x \in \mathbb{R}^n, \ u \in \mathbb{R}^m, \ y \in \mathbb{R}^p \).
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\Sigma : \begin{cases} 
\dot{\hat{x}}(t) = \hat{f}(\hat{x}(t)) + \hat{B}u(t), \\
\hat{y}(t) = \hat{C}\hat{x}(t), \quad \hat{x}(0) = \hat{x}_0,
\end{cases}
\]

with \( \hat{f} : \mathbb{R}^{\hat{n}} \rightarrow \mathbb{R}^{\hat{n}}, \ \hat{B} \in \mathbb{R}^{\hat{n} \times m}, \ \hat{C} \in \mathbb{R}^{p \times \hat{n}}, \ \hat{x} \in \mathbb{R}^{\hat{n}}, \ u \in \mathbb{R}^m, \ \hat{y} \in \mathbb{R}^p, \ \hat{n} \ll n. \)

**Goal**

\( \hat{y} \approx y \) for all admissible \( u. \)
Proper Orthogonal Decomposition (POD)

- Take computed or experimental 'snapshots' of full model: 
  \[ x(t_1), x(t_2), \ldots, x(t_N) \] =: X,
- perform SVD of snapshot matrix: 
  \[ X = VSW^T \approx V_n S_n W_n^T. \]
- Reduction by POD-Galerkin projection: 
  \[ \dot{\hat{x}} = V_n^T f(V_n \hat{x}) + V_n^T Bu. \]
- Requires evaluation of \( f \)
  \( \mapsto \) discrete empirical interpolation [Sorensen/Chaturantabut '09].
- Input dependency due to 'snapshots'!
Proper Orthogonal Decomposition (POD)

- Take computed or experimental 'snapshots' of full model:
  \[x(t_1), x(t_2), \ldots, x(t_N)] =: X,
- perform SVD of snapshot matrix: \(X = VS^TW^T \approx V_nS_nW_n^T\).
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Trajectory Piecewise Linear (TPWL)

- Linearize \(f\) along trajectory,
- reduce resulting linear systems,
- construct reduced model by weighting sum of linear systems.
- Requires simulation of original model and several linear reduction steps, many heuristics.
  \(\leadsto\) talk by J.P. Amorocho, SyreNe dissemination mini-symposium (Thu, 15h)
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Nonlinear Model Reduction

Model Reduction via Bilinear Approximations
- Bilinear Control Systems
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Outlook
Consider continuous-time bilinear systems of the form

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\Sigma_c : \begin{cases} 
\dot{x}(t) = Ax(t) + \sum_{j=1}^{m} N_j x(t) u_j(t) + Bu(t), \\
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\end{cases}
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where \( A, N_j \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \), \( C \in \mathbb{R}^{p \times n} \), \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^m \), \( y \in \mathbb{R}^p \).
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Output Characterization: Volterra series

\[
y(t) = \sum_{j=1}^{\infty} \int_{0}^{t} \int_{0}^{t_1} \cdots \int_{0}^{t_{j-1}} h(t_1, \ldots, t_j) u(t - t_1 - \cdots - t_j) \cdots u(t - t_j) dt_j \cdots dt_1,
\]

with kernels \( h(t_1, \ldots, t_j) = Ce^{At_j} N \cdots e^{At_2} Ne^{At_1} B. \)
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\]

with kernels \( h(t_1, \ldots, t_j) = Ce^{At_j} N \cdots e^{At_2} Ne^{At_1} B. \)

**Multivariable Laplace-transform:**

\[
H(s_1, \ldots, s_j) = C(s_j I - A)^{-1} N \cdots (s_2 I - A)^{-1} N(s_1 I - A)^{-1} B.
\]
Approximate nonlinear state evolution function $f$ by Taylor polynomial, e.g.

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\dot{x} = f(x) + Bu \approx A_1 x + A_2(x \otimes x) + Bu.
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Construct enlarged bilinear system as

$$\frac{d}{dt} \begin{bmatrix} x \\ x \otimes x \end{bmatrix} \approx \begin{bmatrix} A_1 & A_2 \\ 0 & A_1 \otimes I + I \otimes A_1 \end{bmatrix} \begin{bmatrix} x \\ x \otimes x \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 \\ B \otimes I + I \otimes B & 0 \end{bmatrix} \begin{bmatrix} x \\ x \otimes x \end{bmatrix} u + \begin{bmatrix} B \\ 0 \end{bmatrix} u,$$

$$y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ x \otimes x \end{bmatrix}.$$
Common approach: expand $j$-th transfer function about $s_j = \sigma_j$:

$$H(s_1, \ldots, s_j) = \sum_{\ell_1, \ldots, \ell_j = 1}^{\infty} m_\sigma(\ell_1, \ldots, \ell_j) s_1^{\ell_1-1} s_2^{\ell_2-1} \cdots s_j^{\ell_j-1},$$

$$m_\sigma(\ell_1, \ldots, \ell_k) = (-1)^j C (A - \sigma_j I)^{-\ell_j} N \cdots (A - \sigma_2 I)^{-\ell_2} N (A - \sigma_1 I)^{-\ell_1} B.$$

**Theorem**

Given $\Sigma$. Construct $\hat{\Sigma}$ by projection $P = VV^T$, where $V$ is given as a basis of the union of the (block) Krylov subspaces

- $\text{span}\{V^{(1)}\} = \mathcal{K}_{q_1} \left( (A - \sigma_1 I)^{-1}, (A - \sigma_1 I)^{-1} B \right)$,
- $\text{span}\{V^{(j)}\} = \mathcal{K}_{q_j} \left( (A - \sigma_j I)^{-1}, (A - \sigma_j I)^{-1} N V^{(j-1)} \right), \quad j = 2, \ldots, r.$

Then for $j = 1, \ldots, r$ and $\ell_k = 1, \ldots, q_k$ ($k = 1, \ldots, j$):

$$m_\sigma(\ell_1, \ldots, \ell_j) = \hat{m}_\sigma(\ell_1, \ldots, \ell_j).$$
Definition [Zhang/Lam’02]

Assume that the solutions of the generalized Lyapunov equations

\[ AP + PA^T + NPN^T + BB^T = 0, \]
\[ A^T Q + QA + N^T QN + C^T C = 0, \]

associated with a bilinear system \( \Sigma \), exist. Then the \( H_2 \)-norm of \( \Sigma \) is defined as

\[ \| \Sigma \|_{H_2} = \sqrt{CPC^T} = \sqrt{B^T QB}. \]

Question: is it possible to characterize the \( H_2 \)-norm in terms of the generalized transfer functions?
Lemma

Let a bilinear system $\Sigma$ be given and let $\sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$ denote the spectrum of $A$. Then the $\mathcal{H}_2$-norm of $\Sigma$ can be alternatively computed as follows

$$||\Sigma||^2_{\mathcal{H}_2} = \sum_{j=1}^{\infty} \sum_{\ell_j=1}^{n} \cdots \sum_{\ell_1=1}^{n} \Phi_{\ell_1,\ldots,\ell_j} H_j \left(-\lambda_{\ell_1}, \ldots, -\lambda_{\ell_j}\right),$$

where

$$\Phi_{\ell_1,\ldots,\ell_j} = \lim_{s_k \to \lambda_{\ell_k}} H_j(s_1, \ldots, s_j)(s_1 - \lambda_{\ell_1}) \cdots (s_j - \lambda_{\ell_j})$$

denote generalized residues associated with the transfer functions.
Consequently, the $\mathcal{H}_2$-norm of the error system is given as

$$
||\Sigma - \hat{\Sigma}||^2_{\mathcal{H}_2} = \sum_{j=1}^{\infty} \sum_{\ell_1, \ldots, \ell_j}^{n} \Phi_{\lambda_{\ell_1}, \ldots, \lambda_{\ell_j}} \left( H_j(-\lambda_{\ell_1}, \ldots, -\lambda_{\ell_j}) - \hat{H}_j(-\lambda_{\ell_1}, \ldots, -\lambda_{\ell_j}) \right)
$$

$$
+ \sum_{j=1}^{\infty} \sum_{\hat{\ell}_1, \ldots, \hat{\ell}_j}^{\hat{n}} \Phi_{\hat{\lambda}_{\hat{\ell}_1}, \ldots, \hat{\lambda}_{\hat{\ell}_j}} \left( \hat{H}_j(-\hat{\lambda}_{\hat{\ell}_1}, \ldots, -\hat{\lambda}_{\hat{\ell}_j}) - H_j(-\hat{\lambda}_{\hat{\ell}_1}, \ldots, -\hat{\lambda}_{\hat{\ell}_j}) \right)
$$

$\Rightarrow$ Necessary $\mathcal{H}_2$-optimality conditions:

$$
H_j(-\hat{\lambda}_{i_1}, \ldots, -\hat{\lambda}_{i_j}) = \hat{H}_j(-\hat{\lambda}_{i_1}, \ldots, -\hat{\lambda}_{i_j}),
$$

$$
\frac{\partial}{\partial s_k} H_j(-\hat{\lambda}_{i_1}, \ldots, -\hat{\lambda}_{i_j}) = \frac{\partial}{\partial s_k} \hat{H}_j(-\hat{\lambda}_{i_1}, \ldots, -\hat{\lambda}_{i_j}),
$$

for $i_1, \ldots, i_j \leq n$, $k = 1, \ldots, j$ and $j = 1, 2, 3, \ldots$
Theorem

Let $\Sigma$ be a bilinear system. Assume that $V$ and $W$ are given as bases of the unions of the column spaces

\[
V_1 = \left[ (\sigma_1 I - A)^{-1} B, \ldots, (\sigma_q I - A)^{-1} B \right],
\]

\[
W_1 = \left[ (\sigma_1 I - A^T)^{-1} C, \ldots, (\sigma_q I - A^T)^{-1} C \right],
\]

\[
V_j = \left[ (\sigma_1 I - A)^{-1} NV_{k-1}, \ldots, (\sigma_q I - A)^{-1} NV_{k-1} \right],
\]

\[
W_j = \left[ (\sigma_1 I - A^T)^{-1} N^T W_{k-1}, \ldots, (\sigma_q I - A^T)^{-1} N^T W_{k-1} \right],
\]

for $j \leq r$. If $\hat{\Sigma}$ is constructed by the Petrov-Galerkin projection with $P = V(W^T V)^{-1} W^T$, it holds

\[
H_j(s_1, \ldots, s_j) = \hat{H}_j(s_1, \ldots, s_j), \quad j \leq 2r,
\]

\[
\frac{\partial}{\partial s_k} H_j(s_1, \ldots, s_j) = \frac{\partial}{\partial s_k} \hat{H}_j(s_1, \ldots, s_j), \quad j = 1, \ldots, r, \quad k = 1, \ldots, j.
\]
Algorithm 1 Bilinear Iterative Rational Krylov Algorithm (Bilinear-IRKA)

Input: \( A, N, B, C, r, q \)  
Output: \( \hat{A}, \hat{N}, \hat{B}, \hat{C} \)

1: Make an initial selection \( \{\sigma_1, \ldots, \sigma_q\} \).
2: \textbf{while} (change in \( \sigma_i > \epsilon \)) \textbf{do}
3: \hspace{1em} Compute \( V = [V_1, \ldots, V_r] \) and \( W = [W_1, \ldots, W_r] \) \( \in \mathbb{R}^{n \times (q+\cdots+q^r)} \).
4: \hspace{1em} Compute truncated SVD \( V_q \) and \( W_q \) of \( V \) and \( W \).
5: \hspace{1em} \( \hat{A} = (W_q^T V_q)^{-1} W_q^T A V_q \)
6: \hspace{1em} \( \sigma_i \leftarrow -\lambda_i(\hat{A}) \)
7: \hspace{1em} \textbf{end while}
8: \( \hat{N} = (W_q^T V_q)^{-1} W_q^T N V_q \), \( \hat{B} = (W_q^T V_q)^{-1} W_q^T B \), \( \hat{C} = CV_q \)

Remark: Exact interpolation properties are lost due to SVD.
Numerical Examples
A Nonlinear RC Circuit

\[ i = u(t) \]

\[ v_1 \]

\[ v_2 \]

\[ v_{\xi-1} \]

\[ v_{\xi} \]

\[ g(v) = e^{40v} + v - 1, \quad C = 1, \]
\[ \dot{v}(t) = f(v(t), g(v(t))) + Bu(t) \]
\[ y(t) = v_1(t) \]

- state-nonlinear control system
- bilinearization yields system dimension \( \xi + \xi^2 \)
Numerical Examples
A Nonlinear RC Circuit

Transient response for $\xi = 100$ and $u(t) = e^{-t}$
Numerical Examples
A Nonlinear RC Circuit

Relative errors for $\xi = 100$, $r = 2$ and $u(t) = e^{-t}$

- IRKA points, $\hat{n} = 4 + 4 \cdot 4$
- B-IRKA points, $\hat{n} = 21$
Relative errors for \( n = 10^{10} \), \( r = 2 \) and \( u(t) = e^{-t} \)

- IRKA points, \( \hat{n} = 4 + 4 \cdot 4 \)
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Numerical Examples
A Nonlinear RC Circuit

Transient response for $\xi = 100$ and $u(t) = \frac{1}{2}(\cos(\frac{1}{5}\pi t) + 1)$
Numerical Examples
A Nonlinear RC Circuit

Relative errors for $\xi = 100$, $r = 2$ and $u(t) = \frac{1}{2}(\cos\left(\frac{1}{5}\pi t\right) + 1)$
Numerical Examples
A Nonlinear RC Circuit

Relative errors for \( n = 10100, r = 2 \) and \( u(t) = \frac{1}{2}(\cos(\frac{1}{3}\pi t) + 1) \)
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4. Outlook
Let us extend our focus to \textit{quadratic-bilinear} systems of the form

\begin{align*}
\dot{x}(t) &= A_1 x(t) + A_2 x(t) \otimes x(t) + Nx(t)u(t) + Bu(t), \\
y(t) &= Cx(t), \quad x(0) = x_0,
\end{align*}

where $A_1, N \in \mathbb{R}^{n \times n}, A_2 \in \mathbb{R}^{n \times n^2}, B, C^T \in \mathbb{R}^n$. 

Additional quadratic term allows \textit{exact} representations of a large class of nonlinear systems. Increase of state dimension, but significantly less than for Carleman. Transfer function approach will open up Krylov-based reduction techniques.
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Quadratic-Bilinear Control Systems
Quadratic-Bilinearization

Theorem [Gu’09]
Assume that the state equation of a nonlinear system $\Sigma$ is given by

$$\dot{x} = a_0 x + a_1 g_1(x) + \ldots + a_k g_k(x) + Bu,$$

where $g_i(x) : \mathbb{R}^n \to \mathbb{R}^n$ are compositions of uni-variable rational, exponential, logarithmic, trigonometric or root functions, respectively. Then, by iteratively taking Lie derivatives, $\Sigma$ can be transformed into a quadratic-bilinear control system of dimension $N > n$. 
Quadratic-Bilinear Control Systems

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Example

- $\dot{x}_1 = \exp(-x_2) \cdot \sqrt{x_1^2 + 1}$, $\dot{x}_2 = \sin x_2 + u$
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Quadratic-Bilinear Control Systems

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- $z_1 := \exp(-x_2)$, $z_2 := \sqrt{x_1^2 + 1}$, $z_3 := \sin x_2$, $z_4 := \cos x_2$
- $\dot{x}_1 = z_1 \cdot z_2$, $\dot{x}_2 = z_3 + u$
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- $z_1 := \exp(-x_2), \quad z_2 := \sqrt{x_1^2 + 1}, \quad z_3 := \sin x_2, \quad z_4 := \cos x_2$
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Theorem [Gu’09]

Assume that the state equation of a nonlinear system $\Sigma$ is given by

$$\dot{x} = a_0 x + a_1 g_1(x) + \ldots + a_k g_k(x) + Bu,$$

where $g_i(x) : \mathbb{R}^n \to \mathbb{R}^n$ are compositions of uni-variable rational, exponential, logarithmic, trigonometric or root functions, respectively. Then, by iteratively taking Lie derivatives, $\Sigma$ can be transformed into a quadratic-bilinear control system of dimension $N > n$.

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- nonlinear system is assumed to be a series of homogeneous nonlinear subsystems, i.e. response should be of the form

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$$\begin{align*}
\dot{x}_1 &= A_1 x_1 + Bu, \\
\dot{x}_2 &= A_1 x_2 + A_2 x_1 \otimes x_1 + Nx_1 u, \\
\dot{x}_3 &= A_1 x_3 + A_2 (x_1 \otimes x_2 + x_2 \otimes x_1) + Nx_2 u \\
& \vdots
\end{align*}$$
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$$\vdots$$

- Although $i$-th subsystem is coupled nonlinearly to preceding systems, linear systems are obtained if terms $x_j$, $j < i$ are interpreted as pseudo inputs.
In a similar way, a series of generalized \textit{symmetric} transfer functions can be obtained via the growing exponential approach

\[
H_1(s_1) = C \underbrace{(s_1 I - A_1)^{-1} B}_{G_1(s_1)},
\]

\[
H_2(s_1, s_2) = \frac{1}{2!} C \left( (s_1 + s_2) I - A_1 \right)^{-1} \left[ N (G_1(s_1) + G_1(s_2)) + A_2 (G_1(s_1) \otimes G_1(s_2) + G_1(s_2) \otimes G_1(s_1)) \right],
\]

\[
H_3(s_1, s_2, s_3) = \frac{1}{3!} C \left( (s_1 + s_2 + s_3) I - A_1 \right)^{-1} \left[ N(G_2(s_1, s_2) + G_2(s_2, s_3) + G_2(s_1, s_3)) + A_2 (G_1(s_1) \otimes G_2(s_2, s_3) + G_1(s_2) \otimes G_2(s_1, s_3) + G_1(s_3) \otimes G_2(s_1, s_3) + G_2(s_2, s_3) \otimes G_1(s_1) + G_2(s_1, s_3) \otimes G_1(s_2) + G_2(s_1, s_2) \otimes G_1(s_3)) \right].
\]
Let us now focus on the first two transfer functions.

- Similar to the bilinear case, multimoments locally characterize transfer functions.
- In order to match derivatives up to order $q - 1$, we will need the following Krylov spaces:

\[
U = \mathcal{K}_q \left( A^{(\sigma)}, A^{(\sigma)} B \right)
\]

for $i = 1 : q$

\[
W_i = \mathcal{K}_{q-i+1} \left( A^{(2\sigma)}, A^{(2\sigma)} NU_i \right),
\]

for $j = 1 : \min(q - i + 1, i)$

\[
Z_i = \mathcal{K}_{q-i-j+2} \left( A^{(2\sigma)}, A^{(2\sigma)} A_2(U_i \otimes U_j + U_j \otimes U_i) \right),
\]

with $A^{(\sigma)} = (A_1 - \sigma I)^{-1}$ and $U_i$ denoting the i-th column of $U$. 
The reduced system $\hat{\Sigma}$ is obtained by the Galerkin-Projection $P = VV^T$:

\[
\hat{A}_1 = V^T A_1 V \in \mathbb{R}^{\hat{n} \times \hat{n}}, \quad \hat{A}_2 = V^T A_2 V \otimes V \in \mathbb{R}^{\hat{n} \times \hat{n}^2},
\]

\[
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- Challenge of computing $\hat{A}_2$ has to be mastered
  → need for useful approximations.
- In contrast to original system, the above matrices are in general dense
  → computational complexity still reduced?
Numerical Examples
A Nonlinear RC-Circuit

\[ i = u(t) \]

\[ g(v) = e^{40v} + v - 1, \quad C = 1, \]

\[ \dot{v}(t) = f(v(t), g(v(t))) + Bu(t) \]

\[ y(t) = v_1(t) \]

- state-nonlinear control system
- 'clever' transformation only doubles the state dimension of the resulting quadratic-bilinear system
Numerical Examples
A Nonlinear RC-Circuit

Transient response for $\xi = 30$ and $u(t) = e^{-t}$
Bilinear vs Quadratic-Bilinear for $\xi = 30$ and $u(t) = e^{-t}$

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<td>$10^{-2}$</td>
</tr>
</tbody>
</table>

- Carleman Bilinearization $n = 930$
- Quadratic-Bilinearization $n = 60$
Red. Quadratic-Bilinear for $\xi = 3000, \sigma = 1$ and $u(t) = e^{-t}$

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Numerical Examples
A Nonlinear RC-Circuit

Transient response for $\xi = 30$ and $u(t) = \cos(\pi t)$
Bilinear vs Quadratic-Bilinear for $\xi = 30$ and $u(t) = \cos(\pi t)$

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Numerical Examples
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To do:

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- Characterization of quadratic-bilinear systems via asymmetric transfer functions.
- Investigate possible two-sided reduction methods for quadratic-bilinear systems.
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Thank you for your attention!