Model Reduction using Center and Inertial Manifolds

Peter Benner*

Berlin-Braunschweig-Chemnitz Workshop
Recent Advances in Model Reduction
Chemnitz, 27 November 2006

Abstract

We review approaches for dimension reduction of smooth nonlinear dynamical systems. Here, the dimension of the state-space is reduced by projecting the system onto center or (approximate) inertial manifolds. Examples taken from the open literature illustrate several aspects one should be aware of when applying these methods.

1 Introduction to the Center Manifold Approach

In dynamical systems theory, dimension reduction is a commonly used technique to study long-time dynamics or asymptotic behavior. This brief note tries to convey the basic ideas behind this. We omit inputs and outputs as they play no role in the reduction process. Thus we consider a "free" dynamical system of the form

$$\dot{x} = Ax + f(x), \quad A \in \mathbb{R}^{n\times n}, \tag{1}$$

with f being of class C^∞ and $x(t) \in \mathbb{R}^n$ being the state of the dynamical system at time $t \in [0, T]$. The matrix A can be considered as the Jacobi matrix $Dg(0)$ if (1) originally results from a nonlinear autonomous system of the form $\dot{x} = g(x)$. Furthermore, we assume (1) to have an isolated equilibrium at $x = 0$ (implying $f(0) = 0$) and $f(x) = \mathcal{O}(\|x\|^2)$ (which relates f to the higher order terms in a Taylor series expansion about $x = 0$ of g)

Remark 1 *Everything derived here can also be obtained for isolated equilibria at $x^* \neq 0$.*

The center manifold approach is based on assuming the existence of a spectral decomposition of A in the following form:

$$T^{-1}AT = D = \begin{bmatrix} A_s & & \\ & A_c & \\ & & A_u \end{bmatrix}, \tag{2}$$

where $\Lambda(A_s) \subset \mathbb{C}^-$, $\Lambda(A_c) \subset \imath\mathbb{R}$, and $\Lambda(A_u) \subset \mathbb{C}^+$. Here, $\Lambda(M)$ denotes the spectrum of the matrix M, $\imath\mathbb{R}$ is the imaginary axis, and $\mathbb{C}^-, \mathbb{C}^+$ denote the open left and right complex half planes.

*Professur Mathematik in Industrie und Technik, Fakultät für Mathematik, TU Chemnitz, 09107 Chemnitz, Germany; benner@mathematik.tu-chemnitz.de.

Next, let the eigenspaces corresponding to A_s, A_c, and A_u, respectively, be denoted by $\mathcal{E}_s, \mathcal{E}_c$, and $\mathcal{E}_u$, respectively. These are called the *stable*, *center*, and *unstable*, resp. eigenspaces. For now, assume

$$\dim(\mathcal{E}_u) = 0, \quad \dim(\mathcal{E}_c) = r, \quad \dim(\mathcal{E}_s) = n - r.$$

Using the change of coordinates

$$x = T \begin{bmatrix} x_s \\ x_c \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} f_s(x_s, x_c) \\ f_c(x_s, x_c) \end{bmatrix} := T^{-1} f \left(T \begin{bmatrix} x_s \\ x_c \end{bmatrix} \right),$$

we can transform (1) to its *canonical form*

$$\begin{aligned} \dot{x}_s &= A_s x_s + f_s(x_s, x_c) \\ \dot{x}_c &= A_c x_c + f_c(x_s, x_c) \end{aligned} \tag{3}$$

Note that due the above assumptions on f, we have $f_s(0,0) = 0$, $f_c(0,0) = 0$ as well as $Df_s(0,0) = \mathcal{O}$ and $Df_c(0,0) = 0$.

The basis for dimension reduction using center manifolds is the following theorem which can be found, e.g., in [1].

Theorem 2 (Center Manifold Theorem) *Given the dynamical system (3), there exists a smooth local manifold, called the* center manifold,

$$\mathcal{W}_c \equiv \mathcal{W}_c^{\text{loc}}(0) := \{(y, z) \in \mathbb{R}^{n-r} \times \mathbb{R}^r \mid y = h(z) \ \forall \|z\| < \delta, \ h(0) = 0, \ Dh(0) = 0\},$$

(here, h is a diffeomorphism, i.e., a smooth, bijective map with h^{-1} of class C^∞) such that (3) restricted to $\mathcal{W}_c$ can be expressed as

$$\dot{x}_c = A_c x_c + f_c(h(x_c), x_c). \tag{4}$$

Proof. See [1]. ∎

Some remarks are in order

a) If f is of class C^k, then $\mathcal{W}_c$ is C^{k-1}.

b) $\mathcal{E}_c$ is the tangent space of $\mathcal{W}_c$ at $\mathcal{O}$.

c) Solutions of (3) which start on $\mathcal{W}_c$ remain on $\mathcal{W}_c$ for all times, i.e., if $x_s(0) = h(x_c(0))$, then $x_s(t) = h(x_c(t)) \ \forall t > 0$.

d) $\mathcal{W}_c$ attracts all "small norm" solutions of (3) exponentially.

e) $\mathcal{W}_c$ is not unique.

Computation of the center manifold

With $x_s = h(x_c)$ we get $\dot{x}_s = Dh(x_c) \cdot \dot{x}_c$ so that (3) becomes

$$Dh(x_c) \cdot \{A_c x_c + f_c(x_s, x_c)\} = A_s h(x_c) + f_s(h(x_c), x_c)$$

which is equivalent to

$$\mathcal{N}(h) := Dh(x_c)\{A_c x_c + f_c(x_s, x_c)\} - A_s h(x_c) - f_s(h(x_c), x_c) = 0. \tag{5}$$

The center manifold can be approximately computed using (5) and the following theorem, also available in [1].

Theorem 3 *Let $\psi : \mathbb{R}^r \to \mathbb{R}^{n-r}$ be of class C^1 with $\psi(0) = 0$ and $D\psi(0) = 0$. If $\mathcal{N}(\psi(x_c)) = \mathcal{O}(\|x_c\|^q)$ (for $\|x_c\| \to 0$ and $q > 1$), then $\|h(x_c) - \psi(x_c)\| = \mathcal{O}(\|x_c\|^q)$.*

To illustrate this, consider the following example.

Example 4 *Let the dynamical system be given by*

$$\dot{x}_1 = x_1^2 x_2 - x_1^5, \quad \dot{x}_2 = -x_2 + x_1^2.$$

Thus, $(0,0)$ is an equilibrium point and the system is already in canonical form with

$$A = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad x_1 \equiv x_c, \ x_2 \equiv x_s.$$

Recalling that we require $h(0) = 0$ and $h'(0) = 0$, we look for

$$x_2 = h(x_1) = \alpha x_1^2 + \beta x_1^3 + \mathcal{O}(x_1^4),$$

i.e., $q = 4$ in Theorem 3. In order to evaluate (5), we need

$$\begin{aligned} \dot{x}_2 &= h'(x_1) \cdot \dot{x}_1 \\ &= (2\alpha x_1 + 3\beta x_1^2 + \mathcal{O}(x_1^3)) \cdot (x_1^2 \cdot h(x_1) - x_1^5) \\ &= 2\alpha^2 x_1^5 + \{2\alpha(\beta - 1) + 3\alpha\beta\} x_1^6 + \mathcal{O}(x^7) \end{aligned}$$

and

$$\begin{aligned} \dot{x}_2 &= -h(x_1) + x_1^2 \\ &= -\alpha x_1^2 - \beta x_1^3 - \mathcal{O}(x_1^4) + x_1^2 \\ &= (1 - \alpha) x_1^2 - \beta x_1^3 + \mathcal{O}(x_1^4). \end{aligned}$$

Setting $\alpha = 1$ and $\beta = 0$ yields

$$h(x_1) = x_1^2 + \mathcal{O}(x_1^4).$$

Hence, on $\mathcal{W}_c$, we have

$$\dot{x}_c = x_c^4 + \mathcal{O}(x_c^5).$$

This suffices to conclude that the origin is an unstable equilibrium.

Remark 5 *Note that using a projection onto E_c, i.e., replacing $\mathcal{W}_c$ by its tangent space $\operatorname{span}\{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]\}$ would yield $\dot{x}_c = -x_c^5$, suggesting a stable equilibrium at $x = O$!*

Remark 5 shows one of the shortcomings of model reduction methods for nonlinear systems based on (linear) Galerkin projection onto subspaces of $\mathbb{R}^n$.

The existence of a center manifold requires A to have some purely imaginary eigenvalues. We briefly discuss alternatives for dimension reduction if this is not the case in the following section.

2 Dimension Reduction based on Inertial Manifolds

If there is no center manifold, e.g., if $\Lambda(A) \subset \mathbb{C}^-$, then w can use (approximate) inertial manifolds.

Definition 6 *$\mathcal{M} \subset \mathbb{R}^n$ is an* inertial manifold *for (1) if $\mathcal{M}$ is a smooth invariant manifold which attracts all "small norm" solutions of (1) exponentially.*

Determining an inertial manifold requires again a spectral decomposition of A and a separation of "fast" and "slow" modes, analogous to singular perturbation approaches for the solution of differential-algebraic equations (DAEs). Let this separation be given by

$$\begin{aligned} T &= [Y, Z], \quad Y \in \mathbb{R}^{n\times r},\ Z \in \mathbb{R}^{n\times n-r}, \\ x &= Yx_1 + Zx_2, \quad Y^T Z = 0. \end{aligned}$$

Then

$$\dot{x}_1 = Y^T AYx_1 + Y^T f(Yx_1 + Zx_2), \tag{6}$$

$$\dot{x}_2 = Z^T AZx_2 + Z^T f(Yx_1 + Zx_2). \tag{7}$$

In a usual linear Galerkin projection based method we would set $x_2 = 0$ in (6). As already pointed out in Remark 5 above, this may lead to qualitative wrong reduced-order approximations!

Better results can be expected using an inertial manifold if it is accessible: assuming (1) has a low-dimensional inertial manifold defined by $x_2 = h(x_1)$ with h being a local diffeomorphism as in Theorem 2, we can integrate (6) error-free and also obtain the solution of (1) error-free as $x = Yx_1 + Zh(x_1)$.

As inertial manifolds are not known in general, one mostly tries to compute an *approximate inertial manifold (AIM)*

$$x_2 \approx \tilde{h}(x_1) =: \tilde{x}_2.$$

Then one substitutes this in (6) so that $x \approx Yx_1 + Z\tilde{h}(x_1)$.

A simple idea to compute $\tilde{h}$ is to solve (6) by backwards Euler using a few steps of fixed point iteration starting from $\tilde{x}_2^{(0)} = 0$. This yields

$$\tilde{x}_2 = \tilde{h}(x_1) = -\tau(I + \tau Z^T AZ)^{-1} Z^T f(Yx_1)$$

with time step τ which can be chosen as $\tau \approx \frac{1}{\lambda_{r+1}}$.

The resulting method can also be considered as a nonlinear Galerkin method.

The AIM approach has been applied to dimension reduction of a variety of nonlinear dynamical systems, including the Navier-Stokes equations; for a survey and references for further reading see [3]. An example of an AIM model reduction approach applied to control problems can be found in [2].

References

[1] J. Carr. *Applications of Center Manifold Theory*, Springer–Verlag, 1981.

[2] K. Ito and K. Kunisch. Reduced order control based on approximate inertial manifolds. *Linear Algebra Appl.*, 415(2–3):531–541, 2006.

[3] M. Meyer. *Reduktionsmethoden zur Simulation des aeroelastischen Verhaltens von Windkraftanlagen*. Dissertation, TU Braunschweig, Schriftenreihe des Mechanikzentrums, 2003.