Model Reduction of Linear Time-Varying Systems using Multipoint Krylov-subspace Projections

Hossain Mohammad Sahadet

Department of Mathematics
Chemnitz University of Technology.
D-09107 Chemnitz
Germany

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1 Introduction

- What is Model reduction?
- Model reduction based on projection formulations.
- Why Krylov-subspace projection?
  - Process based on ‘moment’ matching.
  - Multipoint rational approximations are efficient for particular cases.
  - Relatively faster than other direct factorization approaches.
The LTI System

The linear time-invariant multi-input, multi-output (MIMO) differential-algebraic system of the form

\[ E \dot{x}(t) = -Ax(t) + Bu(t) \]

\[ y(t) = C^T x \]

where \( x(t) \in \mathbb{R}^n, \ u(t) \in \mathbb{R}^{n_i} (n_i \leq n), \ y(t) \in \mathbb{R}^{n_0}. \)

Transfer function of the LTI System

Applying the Laplace Transforms on the system and assuming \( x_0 = 0 \) gives

\[ \tilde{y}(s) = c^T (sI + A)^{-1} B \tilde{u}(s) \]

where \( \tilde{\ } \) denotes the Laplace transform.

Here \( c^T (sI + A)^{-1} B \) is called the Transfer function of the LTI system.
Suppose that a rational approximation of the LTI system (1) is given by
\begin{align*}
\hat{E} \dot{z}(t) &= -\hat{A} z + \hat{B} u(t) \\
y(t) &= \hat{C}^T z
\end{align*}
where \( \hat{E}, \hat{B} \in \mathbb{R}^{r \times r}, z(t) \in \mathbb{R}^r, \hat{C} \in \mathbb{R}^{r \times n_0} \) and \( r \ll n \).

The reduced matrices are obtained through the projection matrices \( P \) and \( Q \)
\[
\hat{E} = P^T E Q, \quad \hat{A} = P^T A Q, \quad \hat{B} = P^T B, \quad \hat{C} = Q^T C
\]

For \( E = I \), the k-th moment of the transfer function of the reduced model is
\[
C^T A^{-k-1} B = C^T A^{-(k+1)} B
\]

Connect the moments with the projection matrices \( P \) and \( Q \).
Definitions and Theorems

- **Definition 1**
  
  *Krylov subspace* is defined as
  
  \[ \kappa_m(A, b) = \text{span}\{b, Ab, A^2b, \ldots, A^{m-1}b\} \]

  where \( A \in \mathbb{R}^{n\times n} \) and \( b \in \mathbb{R}^n \) is called the *Starting vector*.

- **The first independent basic vectors can be considered as a basis of a Krylov subspace.**

- **Definition 2**
  
  *Block Krylov subspace* is defined as
  
  \[ \kappa_m(A, B) = \text{span}\{B, AB, A^2B, \ldots, A^{m-1}B\} \]

  where \( A \in \mathbb{R}^{n\times n} \) and the columns of \( B \in \mathbb{R}^{n\times m} \) are the *starting vectors*. 
Definitions and Theorems

(Joel R. Phillips’ 1998)

Theorem 1
If the columns of $Q$ form a basis for $\kappa_m(A^{-1}E, A^{-1}B)$ and $P \in \mathbb{R}^{n \times m}$ is chosen such that $\hat{A}$ is non-singular, then the first $m$ moments (about zero) of the original and reduced order system match.

Theorem 2
If the columns of $P$ form a basis for $\kappa_n(A^{-T}E^T, A^{-T}C)$ and the matrix $Q$ is chosen such that $\hat{A}$ is nonsingular, then the first $n$ moments of the original and reduced order system match.

Theorem 3
If the columns of $P$ span $\kappa_m(A^{-T}, C)$, and the columns of $Q$ span $\kappa_n(A^{-1}, B)$, then $\hat{C}^T(sI + \hat{A})^{-1}\hat{B}$ matches the first $m+n$ moments of $C^T(sI + A)^{-1}B$. 

Some projection techniques

- **The projection techniques**
  - **Lanczos process**: $P^T Q = I$ (that means, the two projection bases are bi-orthogonal)
  - **Arnoldi Process**: $P = Q$ and $P^T Q = I$.
  - **Structural properties are inherited.**
Multipoint approximations

- The transfer function and some of its derivatives (moments) are matched at several points in the complex plane.

- $P$ and $Q$ must contain a basis for the union of the Krylov subspace constructed at the different expansion points.

- For a complex expansion point, a real model can be efficiently obtained.

- For a real matrix $A$, if $\tilde{u} = (I - sA)^{-1} p$ is in Krylov subspace, then its conjugate $\tilde{u}^*$ is also in the Krylov subspace.


3 LTV signal analysis

- Consider the nonlinear system describing a circuit equation

\[
\frac{dq(v(t))}{dt} + f(v(t)) = b_1(t) + b u(t)
\]

\[z_1(t) = c^T v(t).
\]  

- split \(v(t)\) and \(u(t)\) into two signal parts

\[u = u^{(L)} + u^{(s)}, \text{ and } v = v^{(L)} + v^{(s)}\]

- linearize around the large signal part \(v^{(L)}\)

\[
G(t)v^{(s)} + \frac{d}{dt} (C(t)v^{(s)}) = b u^{(s)}(t)
\]  

where \(G(t) = \partial f(v^{(L)}(t)) / \partial v^{(s)}\) and \(C(t) = \partial q(v^{(L)}(t)) / \partial v^{(s)}\).
Frequency-domain matrix of LTV system

Follow the formalism of L. Zadeh’s (1950) variable transfer functions and then obtain

\[ v(t) = \int_{-\infty}^{\infty} h(i\omega', t) u(\omega') e^{i\omega' t} d\omega' \]  

(4)

Assume \( u \) is a single response and then using the delta function \( u_{\omega'} = u_{\omega} \delta(\omega - \omega') \) to obtain

\[ v(t) = h(i\omega, t) u(\omega) e^{i\omega t} \]  

(5)

Write \( s = i\omega \) and substitute in (3) to get

\[ G(t)h(s,t) + \frac{d}{dt} (C(t)h(s,t)) + sC(t)h(s,t) = b \]  

(6)
Now defining

\[ K = G(t) + \frac{d}{dt}C(t) \quad \text{and} \quad D = C(t), \]

equation (6) can be written as

\[ [K + sD] h(s, t) = b \]  \hspace{1cm} (7)

Equation (7) is known as the Frequency-domain expression of LTI transfer functions.

- \( h(s, t) \) is a rational function with infinite number of poles.

For example, in a periodically time-varying system with fundamental frequency \( \omega_0 \), if \( \eta \) is a pole of the system, then \( \eta + k\omega_0 \), \( k \) is an integer, will be a pole of \( h(s, t) \).
4 Model reduction of LTV systems

- Obtaining Discrete Rational Functions

Let the LTV system be periodic. We discuss only the SISO case. We seek a representation of transfer functions in terms of finite-dimensional matrices.

- discretization of the operators $\mathcal{K}$ and $\mathcal{D}$.
- use backward-Euler discretization method. Then we get

$$
K = \begin{bmatrix}
\frac{d_1}{h_1} + G_1 & -\frac{d_M}{h_1} \\
-\frac{d_1}{h_2} & \frac{d_2}{h_2} + G_2 & -\frac{d_M}{h_2} \\
& & \ddots & \ddots \\
& & & -\frac{d_{M-1}}{h_M} & \frac{d_M}{h_M} + G_M
\end{bmatrix}
$$

(8)

$$
D = \begin{bmatrix}
d_1 \\
d_2 \\
& \ddots \\
& & d_M
\end{bmatrix}
$$

(9)

$$
h(s) = [h_1(s) \quad h_2(s) \quad \cdots \quad h_M(s)]^T
$$

(10)
Approximating the Krylov subspace

for a small-signal steady state system, we need to solve the finite-difference equations

\[
\begin{bmatrix}
\frac{d_1}{h_1} + G_1 \\
- \frac{d_1}{h_2} \frac{d_2}{h_2} + G_2 \\
\ddots \\
- \frac{d_{m-1}}{h_{m-1}} \frac{d_m}{h_m} + G_m
\end{bmatrix}
\begin{bmatrix}
\tilde{v}(t_1) \\
\tilde{v}(t_2) \\
\vdots \\
\tilde{v}(t_M)
\end{bmatrix} =
\begin{bmatrix}
\tilde{b}(t_1) \\
\tilde{b}(t_2) \\
\vdots \\
\tilde{b}(t_M)
\end{bmatrix}
\]

(11)

where \( \alpha(s) \equiv e^{-sT} \), \( T \) is the fundamental period.

the transfer function then becomes

\[ h(s, t) = e^{-sT} \tilde{v}(t) \]

factorize \( K \) of eqn. (8) into lower and upper triangular parts.
Equation (11) then takes the form

\[(L + \alpha(s)U)\tilde{v}(s) = \tilde{b}(s)\]  \hspace{1cm} (12)

Defining a small-signal modulation operator \(\Psi(s)\).

\[\Psi(s) = \begin{bmatrix} Ie^{st_1} \\ 0 & Ie^{st_2} \\ \vdots & \vdots \\ 0 & Ie^{st_M} \end{bmatrix}\]  \hspace{1cm} (13)

the transfer function has the form

\[h(s) = \Psi^H(s)\tilde{v}(s)\]  \hspace{1cm} (14)

Finally the approximation comes to the form

\[K + sD = \Psi(s)[L + \alpha(s)U]\Psi^H(s)\]  \hspace{1cm} (15)
Approximation by Direct approach

- $L$ is lower triangular, $L^{-1}$ can easily be obtained. Then eqn.(12) takes the form

\[(I + \alpha(s)L^{-1}U) \tilde{v}(s) = L^{-1}\tilde{b}(s)\]  \hspace{1cm} (16)

- In eqn.(16), $L^{-1}U$ has nonzero entries. Hence, $(I + \alpha(s)L^{-1}U)$ has the form

\[
\begin{bmatrix}
I & \cdots & 0 & \alpha(s)R_1 \\
0 & I & \alpha(s)R_2 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0 & I + \alpha(s)R_M
\end{bmatrix}
\]

where $R_j \in \mathbb{R}^{N \times N}$ is the $((j-1)N)+1$ through $jN$ rows of the last columns of $L^{-1}U$.

- **Advantages**: easy to compute, need backsolving technique and simple matrix factorization.

- **Disadvantages**: cost of computation increases very rapidly with the growth of problem size.
The preconditioned problem can be found in the form

\[(I + \alpha(s)L^{-1}U) \tilde{v}(s) = L^{-1} \tilde{b}(s)\]  

Factorization are required only for the diagonal blocks.
Backward substitution are also needed.

**Advantages:**
- reduces the cost.
- easy computations.
- convergence may be faster.

**Disadvantages:**
- reduced model may not preserve the system structure.
Projection by using the finite-difference technique

- Basis for the projection is obtained using finite-difference equations.

Very good approximation to the Krylov-subspaces of spectral operator can be obtained.

The following algorithm shows the detail study.
Algorithm I (Approximating Multipoint Krylov-subspace Model Reduction)

set $k = 1$
for $i = 1, \ldots, n_q$ \{ 
  for $j = 1, \ldots, m(i)$ \{ 
    if $j = 1$ then 
      $w = b$
    else 
      $w = D v_k$
    $u = \Psi_H(s_i)[L + \alpha(s_i)U]^{-1} \Psi(s_i)w$
    for $l = 1, \ldots, k-1$ \{ 
      $u = u - v_l^T u$
    } 
    $v_k = u / \| u \|$
    $k = k + 1$
  } 
\} 

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Algorithm I (Approximating Multipoint Krylov-subspace Model Reduction)

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\[ \hat{K} = V^T K V \]
\[ \hat{E} = V^T E V \]
\[ \hat{C} = V^T C \]
\[ \hat{B} = V^T B \]
Recycled Krylov-subspace scheme

- Assume $E = I$ and take the expansion point at origin. Then exact $V_k$ is constructed so that
  
  $$V_k \subset K_k (K^{-1}, b) \equiv \{b, K^{-1}b, \ldots, K^{-(k-1)}b\}$$

- Every new $u_i$ in $K(K^{-1}, b)$ is related to $K_m(K, b)$ for $i < m$.

- Van der Vorst’87 has shown
  
  $K^{-q}b$ can be constructed from nearly the same Krylov sequence $\{b, Kb, \ldots, K^m b\}$, $m > q$, that is used to construct $K^{-1}b$.
  
  This implies $V_k$ can be constructed from $\{b, Kb, \ldots, K^m b\}$, where $m > k$. 
Advantages:

- no extra cost required to obtain the projectors at multi-frequency points.
- relatively faster convergence.
Applications

- RF circuit simulator
- Switched-Capacitor Synchronous Detector
- Transmitter

Reference:
The End

Thanks for your attention

The full-text of this talk can be provided through mail to mohh@hrz.tu-chemnitz.de