Review of model order reduction methods for numerical simulation of nonlinear circuits

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Abstract

In this paper, we reviewed several newly presented nonlinear model order reduction methods, we analyze these methods theoretically and with experiments in detail. We show the problems exists in each method and future work needs to be done. Besides, we propose the two sided projection method which greatly improved the efficiency of the variational equation order reduction method.

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1. Introduction

The basic idea of model order reduction of a circuit system is to replace the original system by an approximating system with much smaller state-space
Reduced-order modelling is well established for linear circuit systems such as electrical interconnect. The famous work is Padé approximation based moment matching methods such as AWE (asymptotic waveform evaluation) [1], PVL (Padé approximation via Lanczos) [3], a good review paper on linear circuit system model order reduction methods is [2].

For nonlinear systems, a traditional method is to first linearize the system then perform model order reduction on the linear system. But it is well known that this method cannot give good approximation results [5]. The “quadratic reduction method” presented in [5] is based on the idea of approximating the nonlinear system by a quadratic system through dropping the terms of more than two degree in Taylor expansion of the nonlinear term. The bilinearization method proposed in [6] first approximating the original nonlinear system by a bilinear system, then doing order reduction on this approximate bilinear system by using the Volterra series representation of bilinear system in control theory [12]. Variational equation model order reduction method in [7] is based on the variational equation theory in [12]. By this method, the original nonlinear system is changed into several linear systems, then order reduction is done on each linear system. Another reduction method is proposed in [9], which uses the derivatives of the state variable to form an orthogonal projection matrix, and then reduce the original system by projection with this orthogonal matrix. There are also some other nonlinear model order reduction methods such as the trajectory piecewise linear method in [11]. POD (proper orthogonal decomposition) method [10] is widely used in the research of fluid mechanics as well as MEMS.

In this paper we reviewed most of the model order reduction methods for nonlinear circuit systems, and present detailed analysis of each method from both theory and numerical experiments. Then we get some conclusions of each method. In order to have a clear understanding of the nonlinear order reduction methods, in Section 2, we first give a short review of the linear order reduction method based on moment matching which is most popular in model order reduction of linear systems. In Section 3, we review the quadratic reduction method [5]. In Section 4, the bilinearization method [6] is analyzed. In Section 5, the reduction method in [9] is presented. In Section 6, variational equation reduction method [7] is discussed, after showing the problems exist in this method, we propose the two sided projection method which can greatly enhance the efficiency of the variational equation reduction method. Finally, we give our conclusions and further work.

The nonlinear systems that we are concerned with in this paper are of the following form already encountered in [5,6,11].
\[
\frac{dx(t)}{dt} = f(x(t)) + bu(t),
\]
\[
y = c^T x(t).
\]

(1)

\(x \in \mathbb{R}^n\) is often referred to as “state variables”, usually the state-space dimension \(n\) is also called the order of the system. The initial condition is \(x(0) = 0\), \(u = u(t)\) is the input signal, \(y = y(t)\) the output response. For simplicity, we consider only SISO (single input–single output) system, that is the input \(u(t)\) and the output \(y(t)\) are both scalar functions, therefore, \(b \in \mathbb{R}^n\), \(c \in \mathbb{R}^n\).

As an experimental example, we use the nonlinear circuit example in [5].

There is one input current source \(i = u(t)\) flowing into node 1. And our output response is set to be the voltage at node 1. We assume there are total \(N\) nodes in this circuit. The final mathematical model of this circuits is of the form (1), and the order of this system is \(n = 100\). We use the solution of (1) by Matlab function “Odel5s” as the accurate output response. We use different inputs to judge the accuracy of the reduced model.

2. Krylov subspace and moment matching of linear system

In this section the Krylov subspace based moment matching model order reduction methods for linear systems is reviewed which will be useful in later part of this paper. For a linear system given as below:

\[
\frac{dx(t)}{dt} = A(x(t)) + Bu(t),
\]
\[
y = C^T x(t),
\]

(2)

where \(A \in \mathbb{R}^{m \times n}\), \(B \in \mathbb{R}^{m \times r}\), \(C \in \mathbb{R}^{n \times l}\). If \(r = 1\) and \(l = 1\), it is called the single input single output system (SISO); if \(r > 1\) and \(l = 1\), it is called multi-input single output system (MISO), which will be encountered in Section 6, in this case
the input \( u(t) \) is not a scalar function but a vector function, the number of components in \( u(t) \) is just \( r \). Laplace transform of Eq. (2) is

\[
sX(s) = AX(s) + BU(s),
Y(s) = C^T X(s),
\]

where we make use of the initial condition \( x(0) = 0 \), then we have a relation of the Laplace transform of the output with that of the input as

\[
Y(s) = H(s)U(s),
\]

where \( H(s) = C^T (sI - A)^{-1} B = -C^T A^{-1} (I - sA^{-1})^{-1} B \)

is known as the transfer function of the linear system (2). And we can expand \( H(s) \) as

\[
H(s) = -C^T A^{-1} (I + sA^{-1} + s^2 A^{-2} + \cdots) B = -\sum_{i=0}^{\infty} m_i s^i,
\]

where

\[
m_i = C^T A^{-(i+1)} B = C^T A^{-i} (A^{-1} B)
\]

is called the \( i \)th moment \( (i = 0, 1, \ldots) \) of the transfer function.

One kind of moment matching method which is mostly used in model order reduction is the Krylov subspace based projection method, e.g. [4]. This kind of method is based on the idea of performing variable change: \( x \approx Vz, z \in \mathbb{R}^q, q \ll n \) on the original linear system (2), \( V \) is the orthogonal projection matrix that maps the \( n \)-dimensional state-space into a \( q \) dimensional state-space and satisfies \( VT V = I \). Then a linear system which is of much smaller order (or state-space dimension) is derived by multiplying \( VT \) on both sides of the equations

\[
VT A^{-1} V \frac{dz(t)}{dt} = VT Vz(t) + VT A^{-1} Bu(t),
\]

\[
y = C^T Vz(t).
\]

Finally, we get the reduced order system:

\[
\frac{dz(t)}{dt} = \tilde{A}z(t) + \tilde{B}u(t),
\]

\[
\tilde{y} = \tilde{C}^T z,
\]

where \( \tilde{A} = (VT A^{-1} V)^{-1}, \tilde{B} = \tilde{A} V^T A^{-1} B, \tilde{C} = V^T C \). It is clear that the order of this reduced system is \( q \). To be sure that the moments of the transfer function of the reduced system (9) could match those of the transfer function of the original system (1), the projection matrix is constructed as follows.
spancolumn\{V\} = K_q(A^{-1}, A^{-1}B) = \text{span}\{A^{-1}B, A^{-2}B, \ldots, A^{-q}B\}  \tag{10}

satisfying the orthogonal condition \( V^T V = I \), where \text{span}column means that the Krylov subspace \( K_q \) is spanned by the columns of \( V \). Then we have the following theorem.

**Theorem 1.** The first \( q \) moments of the reduced transfer function \( \tilde{H}(s) = -C^T A^{-1} (I - sA^{-1})^{-1} B \) are the same as those of the original transfer function \( H(s) = -C^T A^{-1} (I - sA^{-1})^{-1} B \).

It is a known fact in model order reduction of linear systems (see also [2] and its references).

From Theorem 1, we can see that if we hope to get a more precise reduced system by matching more moments of the transfer function, we have to add more moment vectors into \( K_q \), then the order \( q \) of the reduced system will increase correspondingly. Furthermore, if the linear system is an MISO system, then \( u(t) \) is a vector rather than a scalar function, if there are many components in \( u(t) \), there are many columns in \( B \) accordingly, then the columns in \( K_q \) will be many, and the number of the columns in \( V \) will be large also. This will probably results in that the order \( q \) of the reduced system is not small enough and even near the original system order \( n \). This case will appear in Section 6 and will be discussed further there.

3. The quadratic reduction method

For all the methods discussed below, we assume \( f(x) \) is smooth enough so that it can be expanded into Taylor series, for example

\[
f(x(t)) = Df(0)x + \frac{1}{2}x^T Hf(0)x + \cdots, \tag{11}
\]

where we use of the fact that \( x(t) = 0 \). \( Df(0) \) is Jacobian matrix of \( f \) at \( 0 \), \( Hf(0) \) is the so-called Hessen tensor, for detailed explanations of \( Hf(0) \) see [5].

We first analyze the quadratic reduction method proposed in [5]. If \( f \) was approximated by the first two components of the Taylor expansion above, then the quadratic nonlinear system below is an approximation to the original nonlinear system (1):

\[
\frac{dx(t)}{dr} = Ax + x^T Wx + bu(t),
\]

\[
y(t) = c^T x(t), \tag{12}
\]

where \( A = Df(0) \), \( W = Hf(0) \).
The projection matrix is computed by
\[ \text{spancolumn}\{V\} = \text{span}\{A^{-1}b, A^{-2}b, \ldots, A^{-q}b\} \]
which is inspired by the linear order reduction process in Section 2, then an
reduction process which is also the same as that in Section 2 is performed.
By variable change: \( x \approx Vz \), a reduced system is derived
\[
\frac{dz}{dt} = \hat{A}z + \hat{A}V^T A^{-1}z V^T A V z + \hat{A}V^T b_1 u(t),
\]
\[ y = c^T V z, \tag{13} \]
where \( \hat{A} = (V^T A^{-1} V)^{-1} \).
From the reduction process above we can see that the quadratic method is
more precise than the traditional linearization order reduction method.
Because the quadratic system (12) is a more precise approximation of the origi-
nal system (1) than the simple linearized system:
\[
\frac{dx(t)}{dt} = Ax + bu(t),
\]
\[ y(t) = c^T x(t). \tag{14} \]
However the linear reduction from (12) to (13) is not yet a satisfactory proc-
ess, because it did not contain the nonlinear part of \( f \) in forming the orthogonal
projection matrix \( V \). Furthermore, it is only a heuristic method inspired by the
linear model order reduction process.

4. Bilinearization reduction method

In this section, we give a discussion of the bilinearization reduction method
proposed in [6]. The Taylor expansion of \( f(x) \) can also be rewritten in
Kronecker product of state variable \( x \), in particular,
\[
f(x) = A_1 x + A_2 x \otimes x + A_3 x \otimes x \otimes x + \cdots. \tag{15} \]
When we use the first two terms of (15) to approximate \( f(x) \), we have:
\[ f(x) \approx A_1 x + A_2 x \otimes x. \]
Denote:
\[
x^\circ = \begin{pmatrix} x \\ x \otimes x \end{pmatrix}, \quad b^\circ = \begin{pmatrix} b \\ 0 \end{pmatrix}, \quad c^\circ = \begin{pmatrix} c \\ 0 \end{pmatrix}, \tag{16} \]
\[ A^\circ = \begin{pmatrix} A_1 & A_2 \\ 0 & A_1 \otimes I + I \otimes A_1 \end{pmatrix}, \quad N^\circ = \begin{pmatrix} 0 & 0 \\ b \otimes I + I \otimes b & 0 \end{pmatrix} \tag{17} \]
through not complicated computation [6], an approximate bilinear system of (1) is

\[
\begin{align*}
\frac{dx^\otimes}{dt} &= A^\otimes x^\otimes + N^\otimes x^\otimes u(t) + b^\otimes u(t), \\
y(t) &= c^\otimes^T x.
\end{align*}
\]

(18)

It should be noted that this bilinear system is of much larger state-space dimension than the original nonlinear system (1).

4.1. Volterra series expression of bilinear system

The response of the bilinear system (18) can be expressed in Volterra series [12]:

\[
y(t) = \sum_{n=1}^{\infty} y_n(t)
\]

the degree-n subsystem is given by

\[
y_n(t) = \int_0^t h_n^{(\text{reg})}(t_1, \ldots, t_n) u(t - t_1 - t_2 - \cdots - t_n) \times u(t - t_2 - t_3 - \cdots - t_n) \cdots u(t - t_n) \, dt_1 \cdots dt_n
\]

and the associated regular kernel is given by

\[
h_n^{(\text{reg})}(t_1, \ldots, t_n) = c^\otimes e^{A^\otimes t_1} N e^{A^\otimes t_{n-1}} \cdots N e^{A^\otimes t} b^\otimes \quad (n = 1, 2, \ldots).
\]

(19)

4.2. Multimoments

In the following discussion, we will drop the signal ‘\(\otimes\)’ for simplicity. The multidimensional Laplace domain transfer function of the n-th degree regular kernel (19) of a bilinear system (18) is given by

\[
H_n^{(\text{reg})}(s_1, s_2, \ldots, s_n) = c^T (s_n I - A)^{-1} N (s_{n-1} I - A)^{-1} \cdots (s_2 I - A)^{-1} N (s_1 I - A)^{-1} b.
\]

By expanding \(H_n^{(\text{reg})}(s_1, s_2, \ldots, s_n)\) into series form similar as (6), it is natural to define the multimoments as follows:

\[
m(l_n, \ldots, l_1) = (-1)^n e^T A^{-l_n} N \cdots A^{-l_2} N A^{-l_1} b,
\]

(20)

where \(l_n, \ldots, l_1\) are integers. The projection matrix can be constructed as follows

\[
\text{span column } \{ V^{(1)} \} = K_{q_1}(A^{-1}, b).
\]

(21)

Then each basis \(V^{(j)}, j > 1\) span a block Krylov subspace through letting \(A^{-1}\) operating on \(N\) times the previous basis \(V^{(j-1)}\)

\[
\text{span column } \{ V^{(j)} \} = K_{q_j}(A^{-1}, N V^{(j-1)}).
\]

(22)
The final projection matrix $V$ span a union of the subspaces $\text{span} \{ V^{(j)} \}$

$$\text{span} \{ V \} = \bigcup_{j=1}^{J} \text{span} \{ V^{(j)} \}. \quad (23)$$

4.3. Reduction

Using $x \approx Vz$ in (18), and multiply with $V^T$, we now derive the reduced order bilinear system

$$\frac{d}{dt}z = V^T A V z + V^T N V z u(t) + V^T b u(t),$$

$$y(t) = c^T x. \quad (24)$$

It is considered as the reduced model of (1).

It is showed by the theorem in [6] that the multimoments of the reduced bilinear system (24) match the multimoments of the unreduced bilinear system (18) to a certain degree, which provide a theoretical basis for the precision of the reduced bilinear system. But how to choose the proper number of multimoments (i.e. the decision of $q_1, \ldots, q_j$) to construct the projection matrix $V$ is still an open problem, because small different choice will result in very different simulation results.

The figures below is the results of different choice of $q_1, q_2$, here we only match up to the first degree multimoments and the second degree multimoments. The size of the reduced system is $q_1 + q_2 = 25$. The figures show the error between the output response of the reduced bilinear system and the original nonlinear system (1). Here we use the testing input $u(t) = 1 + \sin(2\pi t) + \sin(10\pi t)$.

We can see from Fig. 1 that when $q_1 = 5, q_2 = 20$, the error is the largest, this means that if there are only small number of the first degree multimoments are matched, the final output response will not precise. From Fig. 2, we also see that when $q_1 = 25, q_2 = 0$, the error is larger than the other two cases, which means that if only the first degree multimoments are matched, the output response is also not precise enough. In fact there is much more choices than the four cases in the figures. So far, there is no basis to decided which choice is the optimal.

In nonlinear circuit analysis, we often need to deal with the following nonlinear system

$$C \frac{dx(t)}{dt} = f(x(t)) + bu(t),$$

$$y(t) = l^T x(t). \quad (25)$$
where the state matrix $C$ is singular. The bilinearization reduction method is hard to solve this kind of equation, since the matrix $C$ cannot be moved from left-hand side of the equation to the right-hand side, thereby it is hard to derive the Volterra series representation of $y(t)$.
5. Derivative matching method

This method is presented in [9] it is based on the time domain Taylor series expansion of $x(t)$ in (1)

$$x(t) = \sum_{k=0}^{\infty} a_k (t - t_0)^k, \quad (26)$$

where $a_k = x^{(k)}(t_0)/k!$, $k = 0, 1, 2, \ldots$, are the normalized time domain derivatives of $x$.

It is shown in [9] that $a_k$ is only precise at the time points near $t_0$, as time marching ahead, new expansion point $t_0$ must be chosen and the derivatives $a_k$ need to be computed once again, it seems that this method has not much difference from the classical time marching method.

6. Variational equation reduction method

In control theory [12], the variational equation approach is a method to derive the various kernels of a nonlinear system. In [7,8] this method is used to do order reduction on a nonlinear system such as (1). The detail of this method is as follows. Consider the response of (1) to the inputs of the form $zu(t)$.

$$\frac{dx(t)}{dt} = f(x) + b(zu),$$

$$y(t) = c^T x(t), \quad (27)$$

where $z$ is an arbitrary scalar. $x(t)$ can be written as an expansion in the parameter $z$ of the form:

$$x(t) = zx_1(t) + z^2 x_2(t) + z^3 x_3(t) + \cdots. \quad (28)$$

Replace $f(x)$ by its Taylor series expansion of Kronecker form (15), then substitute (28) and (15) into the right-hand side of (27), and substitute (28) into the left-hand side of (27), we get

$$z \frac{dx_1(t)}{dt} + z^2 \frac{dx_2(t)}{dt} + z^3 \frac{dx_3(t)}{dt} + \cdots$$

$$= zA_1 x_1 + z^2[A_1 x_2 + A_2 (x_1 \otimes x_1)] + \cdots + b(zu) \quad (29)$$

since this equation holds for all $z$, coefficients of like powers of $z$ can be equated. This gives the variational equations:

$$\frac{dx_1(t)}{dt} = A_1 x_1 + bu(t), \quad (30)$$

$$\frac{dx_2(t)}{dt} = A_1 x_2 + A_2 (x_1 \otimes x_1), \quad (31)$$
\[
\frac{dx_3(t)}{dt} = A_1 x_3 + A_2 (x_1 \otimes x_2 + x_2 \otimes x_1) + A_3 (x_1 \otimes x_1),
\]
(32)

\[
\vdots
\]

The idea of this method is that instead of reducing (1), we only need to reduce the above linear systems (30)–(32). Then \(x(t)\) can be gotten through \(x_k(t)\), \(k = 1, 2, 3, \ldots\), by (28) and the output response \(y(t)\) is known also.

### 6.1. Problem exists in the one sided projection

In [7,8], one sided projection method is used to reduce the order of the linear systems above. For example, for the first linear system (30), a projection matrix \(V_1\) is computed based on \(A_1, b\), such that the columns of \(V_1\) span the Krylov subspace \(K_{q_1} (A_1^{-1}, A_1^{-1}b)\), i.e.

\[
\text{span}\{V_1\} = K_{q_1} (A_1^{-1}, A_1^{-1}b) \\
\equiv \text{span}\{A_1^{-1}b, A_1^{-2}b, \ldots, A_1^{-q_1}b\}.
\]

(33)

For a single input system (30), the number of the columns in \(V_1\) is usually \(q_1\), therefore, through variable change \(x_1 \approx V_1 z_1\), we have \(z_1 \in R^{q_1}\), then the reduced linear system of (30) is

\[
\frac{dz_1}{dt} = \tilde{A}_1 z_1 + \bar{b}u(t),
\]

(34)

where \(\tilde{A}^{-1} = (V_1^T A_1^{-1} V_1)^{-1}\), \(\bar{b} = \tilde{A} V_1^T A_1^{-1} b\), \(\tilde{c} = V_1^T c\).

This reduction process is the same as that in Section 2, therefore, this reduced system (34) satisfy Theorem 1.

From Theorem 1, we see that the precision of the reduced model is directly decided by its order \(q_1\), if \(q_1\) is small, then the precision of the reduced system is not high because only small number of moments of the original transfer function can be matched.

As far as the second linear system (31) and the third linear system (32) are concerned, it is not difficulty to see that these two linear systems are both multi-input systems and their inputs are both decided by \(z_1\). Because, from (34) we have

\[
x_1 \otimes x_1 \approx V_1 z_1 \otimes V_1 z_1 = (V_1 \otimes V_1)(z_1 \otimes z_1)
\]
then the inputs in the second linear system (31) becomes \((z_1 \otimes z_1)\), and the inputs in the third linear system (32) include the term \((z_1 \otimes z_1 \otimes z_1)\) which means the number of the inputs in (32) is at least \(q_1^3\).

So far the problem of the one sided projection is clear. On the one hand if \(q_1\) is not too small, the input number \(q_1^3\) will be large, then as already discussed at
the end of Section 2, it is difficult for us to try to reduce the order of the third linear system (32) to a small size, on the other hand if $q_1$ is small enough for us to reduce the order of (32) to a moderate size, unfortunately the error of the reduced model will be too large to be accepted because only small number of moments of the original transfer function can be matched. It can also be seen from the experimental results below (see Fig. 3 and its explanations, see also Table 1 for the results by other testing inputs).

In the figure below, $q_1$, $q_2$, $q_3$ means that we reduce the three linear systems (30)–(32) to a reduced system of order $q_1$, $q_2$, $q_3$ respectively, $j_1$, $j_2$, $j_3$ means the number of moments that have been matched when we reducing each linear system.

In Fig. 3, we use the sine function $u(t) = 1 + \sin(2\pi t) + \sin(10\pi t)$ as our testing inputs, in one case, we reduce the first linear system (30) to a reduced system of order $q_1 = 6$, if we only match the first moment, when reducing the second linear system (31) (i.e. $j_2 = 1$), we get a reduced system of order $q_2 = 10$ and, in order to reduce the third linear system to a moderate size, we only match the first moment (i.e. $j_3 = 1$), and we have $q_3 = 28$, but from the results corresponding to “$j_1 = 6$, $q_1 = 6$, $j_2 = 1$, $q_2 = 10$, $j_3 = 1$, $q_3 = 28$” in Fig. 3, the output response (crossing line) is far away from the exact solution which is the solid line. On the other hand, if we want to derive a more precise output response, we must increase $q_1$, for example, if we increase $q_1$ to $q_1 = 12$, then we have to reduce the second linear system to $q_2 = 22$, and the third linear system to $q_3 = 65$, which we also only matched the first moment of their transfer function respectively. Although the result is more precise enough (see the

![Fig. 3. Output comparison for variational equation reduction method.](image-url)
dashed line in Fig. 3), the cost is that we must solve a reduced system of order \( q_3 = 65 \), which is close to the original system order \( n = 100 \), so it is not a really reduced system, which clearly makes the whole order reduction process much less efficient.

We list the results of several order reduction methods in Table 1 by using different testing inputs \( u \), which are the classical testing inputs in circuit system simulation.

In Table 1, we define the relative error \( e \) of the approximate solution \( x_a \) to the exact solution \( x \) as \( e = \|x_a - x\|_2/\|x_0 - x\|_2 \), assuming \( x_0 \) is the initial solution. We denote error\(_{qu}\) the relative error of the output response of quadratic order reduction method in Section 3, error\(_{bi}\) the relative error of the output response of bilinearization order reduction method in Section 4, error\(_1\) the relative error of the output response of variational equation order reduction method when \( q_1 = 6, q_2 = 10, q_3 = 28 \) and error\(_2\) corresponding to the variational equation order reduction method when \( q_1 = 12, q_2 = 22, q_3 = 65 \).

In Table 1, we can see that for other testing inputs, error\(_1\) is also much larger than error\(_2\) and is larger than the errors of other methods too.

### 6.2. Two sided projection

From the above analysis, we see that if we can find another way to keep the order of the first reduced linear system (34) to a small size and enhance its precision greatly at the meantime, then we can reduced the order of third linear system (32) to a moderate size and keep its reduced model to a high precision concurrently. In the following, we propose the two sided projection method to meet such request.

Instead of using one projection matrix \( V \) to reduce the first linear system (30), we construct two projection matrices \( V \) and \( W \) such that they satisfy the biorthogonal condition \( W^TV = I \), and the columns of \( V, W \) span the two Krylov subspaces below respectively

\[
\text{span}\{\text{column}\ V\} = K_{q_1}(A_1^{-1}, A_1^{-1}b) \\
\equiv \text{span}\{A_1^{-1}b, A_1^{-2}b, A_1^{-3}b, \ldots, A_1^{-q_1}b\}, \quad (35)
\]
\[ \text{span column}\{W\} = K_{q_1}(A_1^{-T}, c) \]
\[ = \text{span}\{c, A_1^{-T}c, (A_1^{-T})^2c, \ldots, (A_1^{-T})^{q_1-1}c\}. \quad (36) \]

Usually we use Lanczos process [3] to form \( V \) and \( W \).

By using variable change \( x_1 \approx Vz_1 \) on system (30), we have
\[
\begin{align*}
A_1^{-1} \frac{dVz_1}{dt} &= Vz_1 + A_1^{-1}bu(t), \\
y_1 &= c^T Vz_1. 
\end{align*}
\quad (37)
\]

We multiply \( W^T \) rather than \( V^T \) on the equations
\[
\begin{align*}
W^T A_1^{-1} \frac{dVz_1}{dt} &= W^T Vz_1 + W^T A_1^{-1}bu(t), \\
y_1 &= c^T Vz_1. 
\end{align*}
\quad (38)
\]

We have the final reduced system
\[
\begin{align*}
\frac{dz_1}{dt} &= \hat{A}_1 z_1 + \hat{b}u(t), \\
y_1 &= \hat{c}^T z_1, 
\end{align*}
\quad (39)
\]
where \( \hat{A}_1 = (W^T A_1^{-1} V)^{-1}, \hat{b} = \hat{A}_1 W^T A_1^{-1} b. \)

The following theorem can be found in [2,3] and its references.

**Theorem 2.** The first \( 2q_1 - 1 \) moment’s of the reduced transfer function \( \hat{H} = -\hat{c}^T(I - \hat{A}_1^{-1})^{-1} \hat{A}_1^{-1} \hat{b} \) are the same as those of the original transfer function \( H = -c^T(I - sA_1^{-1})^{-1} A_1^{-1} b. \)

Compared with Theorem 1 we can see that if we reduce the first linear system (30) to the same order \( q_1 \) by this two projection methods respectively, the two sided projection method can match much more moments than the one sided projection method. Thus under the precondition of reducing (30) to the same small order, we can get a much more precise reduced model by two sided projection so that we can reduced the third linear system (32) to a moderate size without losing much precision. Experimental results in Table 1 also proved our analysis. In Table 1, error\(_{\text{two}}\) is the relative error of the output response of the two sided projection method, in this case we use two sided projection method on the first linear system (30), and for reducing the order of the second and the third linear system (31) and (32), we still use one sided projection method. By comparing with error\(_{\text{one}}\), we can see that to reduce the three linear systems into the same order \( q_1 = 6, q_2 = 10, q_3 = 28 \) the two sided projection method is much more precise than the merely one sided projection method. And we also see that error\(_{\text{two}}\) is almost comparable with error\(_{\text{two}}\), which means that by using two sided projection method on the first linear system, we can
get a relatively precise solution only by solving a reduced system of order 28 at most rather than a system of order 65, which can enhance the efficiency of the variational equation reduction method greatly.

7. Conclusion

From the analysis of different nonlinear model order reduction methods, we can see that model order reduction of a nonlinear system is much more complicated than that of a linear system. The quadratic method is simple but has not a clearly theoretical basis; the bilinearization method is based on a strong theory background, but the reduction process is more complicated and how to choose proper number of different degree multimoments is still a question. Finally, it cannot deal with the general circuit system (25) so far; although the idea of the variational equation method is elegant, and it can easily deal with such system as (25), the efficiency of this method are limited by its exponentially increasing number of inputs. The two sided projection method we proposed can greatly enhance its efficiency. More work will be done as far as the above problems are concerned.

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