Max Planck Institute Magdeburg
Preprints

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Model Order Reduction for a family of systems with application in parametric and uncertain systems

MPIMD/14-05 **March 18, 2014**

Impressum:

Max Planck Institute for Dynamics of Complex Technical Systems, Magdeburg

Publisher:
Max Planck Institute for
Dynamics of Complex Technical Systems

Address:
Max Planck Institute for
Dynamics of Complex Technical Systems
Sandtorstr. 1
39106 Magdeburg

http://www.mpi-magdeburg.mpg.de/preprints/

1 Introduction

The original motivation for this work arouse from parametric model order reduction (MOR). We are interested in creating good reduced order models with respect to the $\mathcal{H}_2$ norm. It is known that for nonparametric systems [5], the optimal reduced system of order r is created by Hermite interpolation at the mirror images of the reduced poles. We do give the necessary background to this in Section 2. More details can be found in [4, 2, 8]. Given a parametric or uncertain system we are interested in creating a good reduced order model for each parameter in a given parameter domain. Assuming we know the $\mathcal{H}_2$ optimal reduced order model together with the optimal interpolation points at a specific parameter we would like to use this information to create a good reduced order model at nearby parameters. We are looking at model order reduction (MOR) from the point of view of rational interpolation and use the barycentric form for that. The reduced order model in state space formulation is such that the transfer function interpolates given points $\sigma_1, \ldots, \sigma_r$ as well as approximates at certain other points $\xi_1, \ldots, \xi_N$. This idea arose from the quite often encountered situation that we have approximate optimal interpolation points $(\sigma_1, \ldots, \sigma_r)$ together with actual frequency response data of the system $(\xi_1, \ldots, \xi_N)$. We will briefly introduce $\mathcal{H}_2$ optimal MOR in the simplest linear case and describe how to create the rational approximant and the corresponding state space system in Section 3. We will then describe how this leads to a method (Section 4) and show some first rather academic examples in Section 5.

2 Background

Given a large scale stable single input single output dynamical system

$$\Sigma : \begin{cases} E\dot{x} = Ax + Bu \\ y = Cx + D \end{cases} \tag{1}$$

where $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^n$, $C \in \mathbb{R}^{1\times n}$, $D \in \mathbb{R}$ such that the pencil (E, A) has only finite eigenvalues in the left half plane, MOR attempts to find a reduced stable dynamical system:

$$\hat{\Sigma} : \begin{cases} \hat{E}\dot{x} = \hat{A}x + \hat{B}u \\ \hat{y} = \hat{C}x + \hat{D} \end{cases}$$

where $\hat{E}, \hat{A} \in \mathbb{R}^{r\times r}$, $\hat{B} \in \mathbb{R}^r$, $\hat{C} \in \mathbb{C}^{1\times r}$, $\hat{D} \in \mathbb{R}$ and the pencil $(\hat{E}, \hat{A})$ has eigenvalues only in the left half plane. Here $r \ll n$ and the map from the input $u \in \mathcal{L}_2(\mathbb{R}^+)$ to the output y of the original system can be well approximated by the map from input to output $\hat{y}$ of the reduced system. In the frequency domain the input-output behavior is characterized by the transfer function. The transfer function H of the original function and the transfer function $\hat{H}$ are given by

$$H(s) = C(sE - A)^{-1}B + D, \quad H(s) = \hat{C}(s\hat{E} - \hat{A})^{-1}\hat{B} + \hat{D}$$

respectively. These functions are complex-valued proper rational functions defined on the complex plane. Looking at the maximal error of the difference between the true output and the reduced output we get [4]:

$$\sup_{t>0} |y(t) - \hat{y}(t)| \leq \|H - \hat{H}\|_{\mathcal{H}_2} \|u\|_{\mathcal{L}_2},$$

where the $\mathcal{H}_2$-norm is defined as

$$\|H - \hat{H}\|^2_{\mathcal{H}_2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(\imath\omega) - \hat{H}(\imath\omega)|^2 d\omega. \tag{2}$$

The reduced order model we want to find minimizes the $\mathcal{H}_2$-error for a given r. Since

$$H(\imath\omega) - \hat{H}(\imath\omega) \to D - \hat{D}$$

as $\omega \to \infty$ we will take $\hat{D} = D$ in our reduced order modeling. Therefore we can set w.l.o.g. assume $D = 0$ and concentrate on strictly proper rational functions.

$\mathcal{H}_2$ optimal model order reduction

Given a state space system (1) it is known that the $\mathcal{H}_2$-optimal reduced order system of order r satisfies

$$H(\sigma_i) = \hat{H}(\sigma_i), \text{ for } i = 1, \ldots, r \tag{3}$$

$$H'(\sigma_i) = \hat{H}'(\sigma_i), \text{ for } i = 1, \ldots, r, \tag{4}$$

where σ_k are the mirror images of the poles of the reduced order system. Since these are not known a priori we need to compute them by an algorithm. The one we will use is called IRKA [4]. Many extensions and improvements of the basic algorithm exist [8, 2, 1]. We will however use only the basic form.

Given the interpolation points, the reduced order model is typically created by creating projection matrices V and W such that the reduced order system is given by

$$\hat{E} = W^T EV, \; \hat{A} = W^T AV, \; B = W^T B, \; C = CV.$$

The projection matrices need to be picked such that $(\sigma I - A)^{-1} B \in \text{Ran}(V)$ and $(\bar{\sigma} I - A^T)^{-1} C^T \in \text{Ran}(W)$ [3], where Ran denotes the range of a matrix. However given $\sigma_k, H(\sigma_j), H'(\sigma_k)$ we can also write down a state space system more directly. This is related to the Loewner framework of reduced order modeling and explained in the following.

Loewner framework

Given frequencies together with the value of the transfer function at those frequencies a data driven approach to MOR is to create a state space system which interpolates there. [7, 6]. Given interpolation points $(\xi_1, \ldots, \xi_r, \sigma_1, \ldots, \sigma_r)$, and its transfer function values:

$$W = [H(\sigma_1), \ldots, H(\sigma_r)], \quad V^T = [H(\xi_1), \ldots, H(\xi_r)],$$

we can define the Loewner matrices

$$\begin{aligned} \mathbb{L}_{ij} &= \frac{V_i - W_j}{\xi_i - \sigma_j} \\ \sigma\mathbb{L}_{ij} &= \frac{\xi_i V_i - \sigma_j W_j}{\xi_i - \sigma_j} \end{aligned} \tag{5}$$

and the order r reduced state space system that interpolates is then given by

$$\hat{E} = -\mathbb{L}, \hat{A} = -\sigma\mathbb{L}, \hat{B} = V, \hat{C} = W,$$

Furthermore if we want to Hermite interpolate σ_i we can create the symmetric Loewner matrices and can create a state space system that Hermite interpolates at $\sigma_1, \ldots, \sigma_r$

$$\hat{E} = -\mathbb{L}, \hat{A} = -\sigma\mathbb{L}, \hat{B} = W^T, \hat{C} = W,$$

where

$$\mathbb{L}_{ij} = \begin{cases} \dfrac{W_i - W_j}{\sigma_i - \sigma_j} \text{ if } i \neq j \\ H'(\sigma_i) \text{ if } i = j \end{cases} \quad \sigma\mathbb{L}_{ij} = \begin{cases} \dfrac{\sigma_i W_i - \sigma_j W_j}{\sigma_i - \sigma_j} \text{ if } i \neq j \\ H(\sigma_i) + \sigma_i H'(\sigma_i) \text{ if } i = j \end{cases}. \tag{6}$$

3 Rational Interpolation

In the following we will develop a reduced order state space system of dimension r whose transfer function is strictly proper and interpolates a given transfer function at points $\sigma_1, \ldots, \sigma_r$. Furthermore the reduced order transfer function is a good approximation on N other points $\xi_1, \ldots, \xi_N \in \mathbb{C}$. If $N = r$ the Loewner matrix approach discussed in the previous section tells us how to create a state space system which interpolates $\sigma_1, \ldots \sigma_r, \xi_1, \ldots, \xi_r$. In the notation above the reduced order transfer function of that system can be written as

$$\tilde{H} = W(\sigma\mathbb{L} - s\mathbb{L})^{-1}V. \tag{7}$$

We will now look at this problem as a rational interpolation problem from the setup of barycentric interpolation. Here we know that the function

$$\tilde{G}(s) = \frac{\sum_{k=1}^{r} \frac{\alpha_k W_k}{s - \sigma_k}}{\sum_{k=1}^{r} \frac{\alpha_k}{s - \sigma_k} + 1}. \tag{8}$$

is a strictly proper rational function that interpolates H at σ_k for all $\alpha_1, \ldots, \alpha_k$. The function value of the transfer function at σ_k is denoted by W_k just as in the definition of the reduced order system via the Loewner matrix approach. We are interested in the choice of α such that $\tilde{G}$ and $\tilde{H}$ are identical.

Lemma 1. *The two transfer functions $\tilde{H}$ and $\tilde{G}$ as in (7) and (8) are identical exactly when*

$$\mathbb{L}\alpha + V = 0$$

Proof. We show that the condition is sufficient. For that we can write both transfer function as sums:

$$\tilde{H}(s) = \sum_{k=1}^{r} W_k \left((\sigma\mathbb{L} - s\mathbb{L})^{-1}V\right)_k, \quad \tilde{G}(s) = \sum_{k=1}^{r} W_k \frac{\frac{\alpha_k}{s-\sigma_k}}{\sum_{k=1}^{r}\frac{\alpha_k}{s-\sigma_k} + 1}.$$

They are then identical if

$$(\sigma\mathbb{L} - s\mathbb{L})^{-1}V = \frac{1}{\sum_{k=1}^{r}\frac{\alpha_k}{s-\sigma_k} + 1}\begin{bmatrix}\frac{\alpha_1}{s-\sigma_1}\\ \vdots \\ \frac{\alpha_r}{s-\sigma_r}\end{bmatrix},$$

which means the following must hold:

$$V = \frac{1}{\sum_{k=1}^{r}\frac{\alpha_k}{s-\sigma_k} + 1}(\sigma\mathbb{L} - s\mathbb{L})\begin{bmatrix}\frac{\alpha_1}{s-\sigma_1}\\ \vdots \\ \frac{\alpha_r}{s-\sigma_r}\end{bmatrix}.$$

The ith component of the right hand side is given by

$$\begin{aligned}
\left(\frac{\sigma\mathbb{L} - s\mathbb{L}}{\sum_{k=1}^{r}\frac{\alpha_k}{s-\sigma_k} + 1}\begin{pmatrix}\frac{\alpha_1}{s-\sigma_1}\\ \vdots \\ \frac{\alpha_r}{s-\sigma_r}\end{pmatrix}\right)_i &= \frac{\sum_{k=1}^{r}\frac{\xi_i V_i - \sigma_k W_k - sV_i + sW_k}{\xi_i - \sigma_k}\frac{\alpha_k}{s-\sigma_k}}{\sum_{k=1}^{r}\frac{\alpha_k}{s-\sigma_k} + 1} \\
&= \frac{1}{\sum_{k=1}^{r}\frac{\alpha_k}{s-\sigma_k} + 1}\sum_{k=1}^{r}\alpha_k\left(\frac{V_i}{s-\sigma_k} - \frac{V_i - W_k}{\xi_i - \sigma_k}\right) \\
&= \frac{\sum_{k=1}^{r}\frac{\alpha_k}{s-\sigma_k}V_i - (\mathbb{L}\alpha)_i}{\sum_{k=1}^{r}\frac{\alpha_k}{s-\sigma_k} + 1} = V_i - \frac{(\mathbb{L}\alpha)_i + V_i}{\sum_{k=1}^{r}\frac{\alpha_k}{s-\sigma_k} + 1},
\end{aligned}$$

which shows that $\tilde{H} = \tilde{G}$ if $\mathbb{L}\alpha + V = 0$. Necessity of the condition follows from the fact that if $\tilde{G} = \tilde{H}$ then we have that $\tilde{G}(\xi_i) = V_i$. This leads to

$$-\sum_{k=1}^{r}\frac{\alpha_k W_k}{\xi_i - \sigma_k} + \sum_{k=1}^{r}\frac{\alpha_k V_i}{\xi_i - \alpha_k} + V_i = 0$$

for all i which then leads to $\mathbb{L}\alpha + V = 0$ Therefore the two rational functions are identical if and only if $\mathbb{L}\alpha + V = 0$. □

We are now interested in the case where $N > r$ and we want to create a state space system of order r. The strictly proper rational function in barycentric form that interpolates all σ_k is given by equation (8). We want to pick α such that $\tilde{G}(\xi_i) \approx H(\xi_i)$. Equality is not possible in general as $N > r$. Looking at the difference we get that

$$\tilde{G}(\xi_i) - H(\xi_i) = \frac{\sum_{k=1}^{r} \frac{\alpha_k W_k}{\xi_i - \sigma_k}}{\sum_{k=1}^{r} \frac{\alpha_k}{\xi_i - \sigma_k} + 1} - V_i = \frac{\sum_{k=1}^{r} \alpha_k \frac{W_k - V_i}{\xi_i - \sigma_k} - V_i}{\sum_{k=1}^{r} \frac{\alpha_k}{\xi_i - \sigma_k} + 1}.$$

Optimizing this is difficult and we therefore are only interested in trying to make the numerator as small as possible. Collecting all numerators in a vector leads to the following vector,

$$(-\mathbb{L}\alpha - V),$$

where $\mathbb{L} \in \mathbb{R}^{N \times r}$ is the Loewner matrix (5), being a rectangular matrix. We will now consider minimizing the 2-norm of this vector.

Lemma 2. *The solution to*

$$\min_{\alpha} f(\alpha) = \|\mathbb{L}\alpha + V\|_2^2$$

is given by the solution to

$$\mathbb{L}^*\mathbb{L}\alpha + \mathbb{L}^*V = 0. \tag{9}$$

Proof. The function to optimize is a quadratic function given by

$$f(\alpha) = \alpha^*\mathbb{L}^*\mathbb{L}\alpha + V^*\mathbb{L}\alpha + \alpha^*\mathbb{L}V + V^*V$$

Setting its derivative equal to 0 results in

$$2\mathbb{L}^*\mathbb{L}\alpha + 2\mathbb{L}^*V = 0$$

□

This means we are interested in a reduced order state space system of dimension r whose transfer function is given by (8), where α solves a linear least squares problem. We will use the Loewner matrix concept but we need to truncate the Loewner matrix. This is done by using the singular value decomposition.

Lemma 3. *A state space system that has the transfer function $\tilde{G}$ as in (8) with $\mathbb{L}^*\mathbb{L}\alpha + \mathbb{L}^*V = 0$ is given by:*

$$\tilde{E} = -Z^*\mathbb{L}, \tilde{A} = -Z^*\sigma\mathbb{L}, \tilde{C} = W, \tilde{B} = Z^*V$$

where $Z = Y(:, 1 : r)$ denotes the first r singular vectors of $\mathbb{L}$ and $\mathbb{L} = Y\Theta X^*$ is its singular value decomposition.

Proof. We have

$$\tilde{H}(s) = W(Z^*\sigma\mathbb{L} - sZ^*\mathbb{L})^{-1}Z^*V = \sum_{k=1}^{r} W_k \left((Z^*\sigma\mathbb{L} - sZ^*\mathbb{L})^{-1}Z^*V\right)_k$$

and $\tilde{G}(s) = \sum_{k=1}^{r} W_k \frac{\frac{\alpha_k}{s-\sigma_k}}{\sum_{k=1}^{r} \frac{\alpha_k}{s-\sigma_k}+1}$. Therefore we need to show

$$(Z^*\sigma\mathbb{L} - sZ^*\mathbb{L})^{-1} Z^* V = \frac{1}{\sum_{k=1}^{r} \frac{\alpha_k}{s-\sigma_k} + 1} \begin{bmatrix} \frac{\alpha_1}{s-\sigma_1} \\ \vdots \\ \frac{\alpha_r}{s-\sigma_r} \end{bmatrix}.$$

We multiply with the inverse matrix and consider the right hand side:

$$\frac{Z^*\sigma\mathbb{L} - sZ^*\mathbb{L}}{\sum_{k=1}^{r} \frac{\alpha_k}{s-\sigma_k} + 1} \begin{bmatrix} \frac{\alpha_1}{s-\sigma_1} \\ \vdots \\ \frac{\alpha_r}{s-\sigma_r} \end{bmatrix} = \frac{Z^*}{\sum_{k=1}^{r} \frac{\alpha_k}{s-\sigma_k} + 1} \left(\sum_{k=1}^{r} \frac{\alpha_k}{s-\sigma_k} V - \mathbb{L}\alpha \right) \tag{10}$$

$$= Z^*V - \frac{Z^*\mathbb{L}\alpha + Z^*V}{\sum \frac{\alpha_k}{s-\sigma_k} + 1} \tag{11}$$

Left to show that $Z^*\mathbb{L}\alpha + Z^*V = 0$. This follows from the fact that $\mathbb{L} = Y\Theta X^* = Z\Theta^r X^*$ where Θ^r is the top r block of the matrix Θ. This equality holds since the rest of the Θ matrix is 0 by definition of the singular value decomposition. From the definition of α we get

$$\begin{aligned} 0 = \mathbb{L}^*\mathbb{L}\alpha + \mathbb{L}^*V = X\Theta^r Z^*\mathbb{L}\alpha + X\Theta^r Z^*V \\ \Rightarrow Z^*\mathbb{L}\alpha + Z^*V = 0 \end{aligned}$$

□

This gives us a recipe on how to create a state space system that satisfies interpolation at $\sigma_1, \ldots, \sigma_r$ and approximation at $\xi_1, \ldots, \xi_N$.

4 Model Order Reduction Method

The results of the previous section, mainly Lemma 3, lets us construct reduced order models. In general the problem setup where this could be useful is when we have several systems which are related and must be reduced. We assume that we know the $\mathcal{H}_2$-optimal interpolation points of one of the systems. The idea is to create reduced order models as above, where the $\sigma_1, \ldots, \sigma_r$ are those known optimal interpolation points and the ξ_i some other chosen values. Then Algorithm 1, which is based on Lemma 3, creates one reduced order system. It needs $V_i = H(\xi_i) - D$ and $W_k = H(\sigma_k) - D$ the value of the strictly proper part of the true transfer function at the approximation/interpolation points.

We compare the reduced order model created by this Algorithm with the system that Hermite interpolates at $\sigma_1, \ldots, \sigma_r$. This is done as described in Section 2. Since we are interested in creating systems whose $\mathcal{H}_2$ error is small the D-term of the original system and reduced system must be the same. Therefore it is enough to just consider the strictly proper part.

Algorithm 1 Calculate $\hat{A}, \hat{B}, \hat{C}, \hat{E}$ via best rational approximation

Require: $r, \xi_1, \dots \xi_N, \sigma_1 \dots, \sigma_r$
Ensure: $\hat{A}, \hat{B}, \hat{C}, \hat{D}, \hat{E}$
  for $i = 1 : N$ **do**
    $V_i = H(\xi_i)$
    for $j = 1 : r$ **do**
      $W_j = H(\sigma_j)$
      $\mathbb{L}_{ij} = \frac{V_i - W_j}{\xi_i - \sigma_j}$
      $\sigma\mathbb{L}_{ij} = \frac{\xi_i V_i - \sigma_j W_j}{\xi_i - \sigma_j}$
    end for
  end for
  Compute the SVD of $\mathbb{L} = U \Sigma V^T$
  $Z = U(:, 1 : r)$
  $\hat{A} = -Z^* \mathbb{L}$
  $\hat{E} = -Z^* \sigma\mathbb{L}$
  $\hat{B} = W$
  $\hat{C} = Z^* V$

The basic idea behind this Moder Order Reduction method is the observation that in many parametric problems, the optimal interpolation points for one parameter are satisfactory interpolation points for another parameter. We therefore realized that in order to create a reduced order model faster we should use these interpolation points. In order to make sure that we are however not creating a large error we use another larger set of points which should be also approximated. We take the ideas from classical data driven MOR and combine those two ideas.

5 Numerical Results

We have two main applications in mind in which approximate information of the $\mathcal{H}_2$-optimal interpolation points are given or could be computed. One is a large scale parametric system for which we want to compute reduced order systems at several parameter values, and the other are systems in which one or more of the matrices are uncertain. In the following we show academic examples that illustrate how to use this MOR technique as well as its benefits and problems.

Example 1: parametric system

The parametric model used in the following is a test problem from the benchmark collection http://www.modelreduction.org, where it is called 'Synthetic parametric model'. It is a system that depends on one parameter and the matrices are created by the MATLAB® function given below for a parameter p.

```
function [A,B,C,D,E]=Para_Model(p)

```

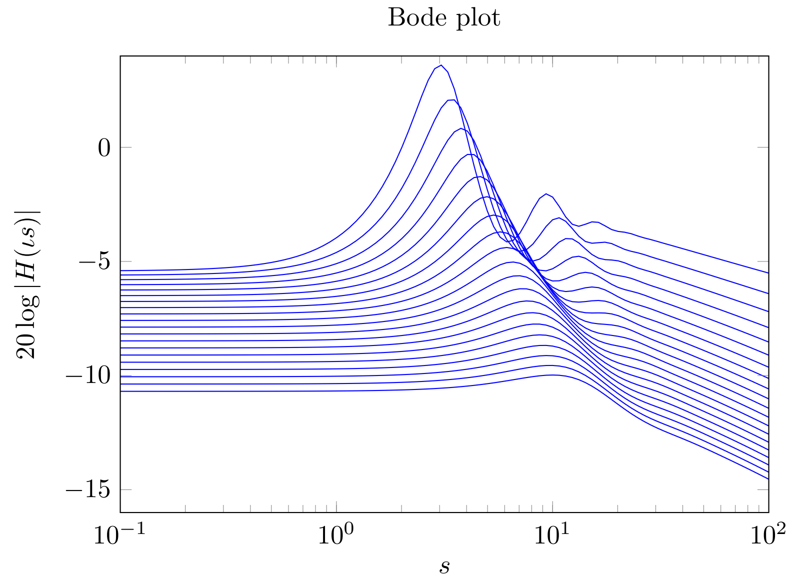

Figure 1: frequency response

```
n = 100;
a = -p*linspace(1e1,1e3,n/2).';     b = p*linspace(1e1,1e3,n/2).';
c = ones(n/2,1);                    d = zeros(n/2,1);
aa(1:2:n-1,1) = a;                  aa(2:2:n,1) = a;
bb(1:2:n-1,1) = b;                  bb(2:2:n-2,1) = 0;
Ae = spdiags(aa,0,n,n);
A0 = spdiags([0;bb],1,n,n) + spdiags(-bb,-1,n,n);
B = 2*sparse(mod([1:n],2)).';
C(1:2:n-1) = c.';                   C(2:2:n) = d.';    C = sparse(C);

A=A0+p*Ae;
D=0;
E=eye(n);
```

The D-term is 0, and as discussed before, and so is $\hat{D}$. The behavior of this model can be seen in Figure 1 which plots the Bode plot of the transfer function for several values of the parameter between 0.3 and 1. This will also be the parameter range of interest in our numerical tests. We consider the system created with parameter value $p = 1$ as the reference system. For that system we compute the $\mathcal{H}_2$ optimal interpolation points $\sigma_1, \ldots, \sigma_r$ with IRKA [4]. We furthermore choose 100 points $\xi_1, \ldots \xi_{100}$, where $\log_{10} \xi_i = -1 + 4i/100$. For a given parameter we create now a reduced order system via Algorithm 1. We also create a state space system that Hermite interpolates at $\sigma_1, \ldots, \sigma_r$. This means at $p = 1$ this system is the $\mathcal{H}_2$-optimal one. However the further away from $p = 1$ we are, the more likely the approximation will not be accurate. In Figure 2(left) we see the Bode plots of the two different approximations for different values of the parameter together with the true transfer function values. Figure 2(right) shows the $\mathcal{H}_2$-error of the two algorithms as well as the $\mathcal{H}_2$ error for each parameter

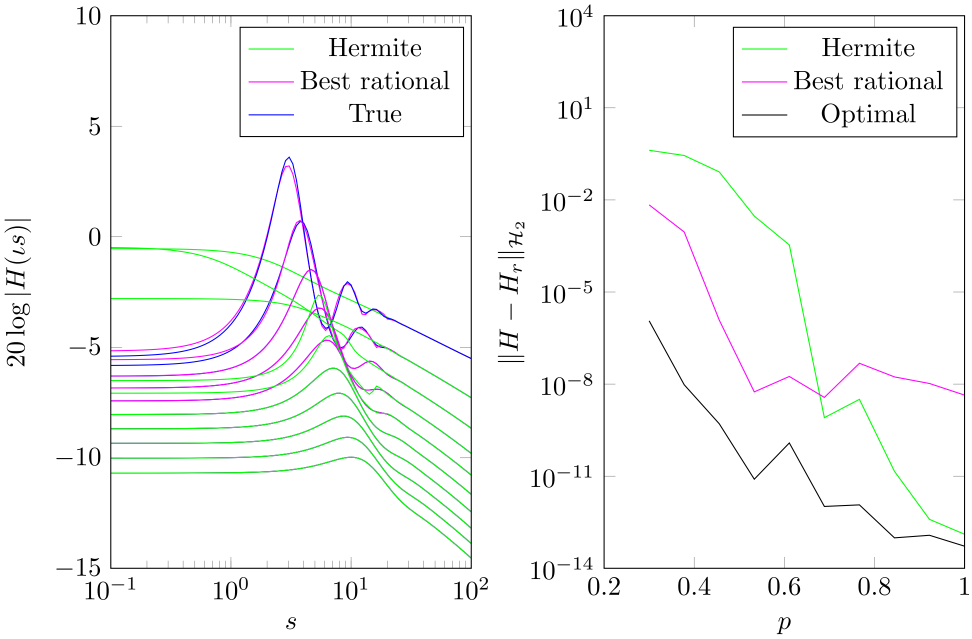

Figure 2: frequency response and error plot

if we compute the $\mathcal{H}_2$-optimal reduced order system via IRKA for each individual parameter. As expected, the Hermite interpolation is only acceptable close to $p = 1$. The reduced order model created by Algorithm 1 follows a similar behavior as the optimal one, but we lose a few orders of magnitude. The time needed to create the reduced order models is especially great for large systems dominated by the number of system solves. In order to create one reduced order model with Algorithm 1, we need to solve $r + N = 120$ large systems. In order to create the optimal reduced order model we need 2r times the number of IRKA steps many system solves. In this example, the average amount of steps needed was around 20 with a few outliers. This results in the total number of $2 * r * 20 = 800$, which is about seven times as many.

Example 2: Uncertainty Quantification

As these numerical experiments attempt to show the features of the algorithm more than real application, we created a demonstration for uncertainty quantification

```
function [A,B,C,D,E]=UQ_example(Q)

n=100;
A=Q*diag(-10*rand(n,1))*Q';
B=ones(n,1);
C=ones(1,n);
D=0;
E=eye(n);
```

We picked Q to be a random orthonormal matrix, however once for the entire test series. A, however, is an uncertain matrix that has many realizations. We compute the $\sigma_1, \ldots, \sigma_r$, the $\mathcal{H}_2$-optimal interpolation points of one realization for different r and choose $\xi_1, \ldots, \xi_N$ such that $\log_{10} \xi_i = -2 + 10i/N$. Given this setup we create for 5 different realizations generated at random a reduced order model via IRKA directly, via Algorithm 1, via Hermite interpolation at the σ_i, via the Loewner matrix approach using also Algorithm 1 with $\sigma_i = \xi_i$ and $\xi_i = \xi_{r+i}, i = 1, \ldots L - r$. In Table 1 we compare the $\mathcal{H}_2$ error between the true and the reduced model as well as the number of large system solves used to create the reduced order model. The Loewner approach in the last column does not always result in a stable reduced order model. The number in brackets shows how often it results in a stable solution out of 5 runs. To get an acceptable approximation we can create a reduced order system 10 times faster than via direct usage of IRKA. The approach is typically better than just a Hermite interpolation of the given σ_k and also the pure interpolation of the ξ_i by a Loewner matrix approach, which also often results in unstable systems. Stability can not be guaranteed in our approach either but often results in stable systems, and in this test example always.

6 Discussion and Conclusions

We have shown an easy and fairly cheap way to create a dimension r state space system that interpolates at r given points and best approximates at N others, where $N > r$. The interpolation points are typically some approximated optimal $\mathcal{H}_2$-interpolation

r	N	IRKA		Algo 1		Hermite		Loewner		
		err	#	err	#	err	#	err	#	
4	8	0.01	142	0.02	12	0.02	8	10	(5)	8
4	16	0.01	142	0.01	20	0.02	8	0.2	(5)	16
6	12	2E-5	134	5E-5	18	2E-4	12	15	(4)	12
6	12	2E-5	134	3E-5	30	2E-4	12	4E-3	(5)	24
10	20	2E-11	151	2E-10	30	2E-5	20	–	(0)	20
10	40	2E-11	151	1E-10	50	2E-5	20	2E-6	(1)	40

Table 1: Error comparison: The optimal $\mathcal{H}_2$ error is compared (for 5 trials or the number in brackets if not all solutions resulted in stable models) to the $\mathcal{H}_2$ error of our method, a hermite interpolation at the approximate system or a Loewner approach of the N given points. Furthermore the number of large system solved required to create the reduced order model is compared.

points. This as shown in the numerical examples gives a technique for parametric model order reduction that performs well in the $\mathcal{H}_2$ norm. We furthermore show how to use the method on state space system with uncertainty. Even though we do lose accuracy compared to the best reduced order model, we believe that this method has potential. It is faster than other methods, and it can be used even if the state space cannot be created easily, whereas measurements are available. And it would then outperform a system created from measurements alone every time. Plus we are more likely to create a stable system. More work is needed to investigate conditions for stability of the reduced system, though. The development of a MIMO version is possible, but technically more involved and not possible in this brief note. This will be reported in the future elsewhere.

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