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Structure Preserving MOR of Large Sparse Second Order Index-1 Systems and Application to a Mechatronics Model

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Abstract

Nowadays machine engineers heavily depend on mathematical models for simulation, optimization and controller design. In either of these tasks reduced dimensional formulations are obligatory in order to achieve fast and accurate results. Usually, the structural mechanical systems of machine tools are of second order type. However, they become descriptor systems when extra constraints are imposed to the systems. This article discusses efficient techniques of Gramian based model order reduction for second order index-1 descriptor systems. The primary focus is on second order to second order reduction techniques for such systems, where the stability of the system can be preserved. The methods are applied to a structural FEM model of a micro-mechanical piezo-actuators based adaptive spindle support (ASS). Numerical results illustrate the efficiency of the techniques.

1 Introduction

This article discusses efficient techniques for model order reduction (MOR) of large-sparse second order index-1 descriptor systems. Mainly, we focus on second order to second order balancing of such systems, so that we can preserve the structure of the original model. We consider the second order systems of the form

$$\begin{aligned}\underbrace{\begin{bmatrix} M_1 & 0 \\ 0 & 0 \end{bmatrix}}_{\mathcal{M}} \begin{bmatrix} \ddot{z}(t) \\ \ddot{\varphi}(t) \end{bmatrix} + \underbrace{\begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix}}_{\mathcal{D}} \begin{bmatrix} \dot{z}(t) \\ \dot{\varphi}(t) \end{bmatrix} + \underbrace{\begin{bmatrix} K_{11} & K_{12} \\ K_{12}^T & K_{22} \end{bmatrix}}_{\mathcal{K}} \begin{bmatrix} z(t) \\ \varphi(t) \end{bmatrix} &= \underbrace{\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}}_{\mathcal{H}} u(t), \\ y(t) &= \underbrace{\begin{bmatrix} B_1^T & B_2^T \end{bmatrix}}_{\mathcal{H}^T} \begin{bmatrix} z(t) \\ \varphi(t) \end{bmatrix} + D_a u(t),\end{aligned} \tag{1}$$

where $z(t) \in \mathbb{R}^{n_1}$, $\varphi(t) \in \mathbb{R}^{n_2}$ are the states, $\mathcal{M}, \mathcal{D}$ and $\mathcal{K} \in \mathbb{R}^{n \times n}$ are the finite element method (FEM)-matrices, $\mathcal{H} \in \mathbb{R}^{n \times p}$ is the input matrix and the output matrix is $\mathcal{H}^T$, i.e., we assume collocated actuators and sensors. The corresponding control input and measurement output to the system are respectively denoted by $u(t)$ and $y(t)$. The matrix $D_a \in \mathbb{R}^{m \times p}$ represents the direct feedthrough from the input to the output. The matrices $\mathcal{M}, \mathcal{D}, \mathcal{K}$, and $\mathcal{H}$ are sparse. We assume the block matrix K_{22} to be nonsingular. We call (1) an index-1 system due to the analogy to first order index-1 (see the next section) linear time-invariant (LTI) systems [29]. The dynamical system (1) usually arises in different branches of engineering such as mechanics [11], where an extra constraint is imposed in order to control the dynamic behavior of the systems, or mechatronics where mechanical and electrical components are coupled with each other. In the specific case of the model example we use in the numerical experiments, the index-1 character results from the multiphysics application with very different timescales. This allows to treat one variable by a stationary analysis, while the other is covered fully dynamic.

If the model is very large, performing the simulation with it has prohibitively expensive computational effort, or is simply impossible due to the limited computer memory.

Therefore, reducing the size of the system is unavoidable for fast simulation. A classical approach to find a reduced order model (ROM) of second order index-1 descriptor systems is first to rewrite (1) in first-order form, then model order reduction (MOR) techniques are applied to find a reduced first-order state space system [29, 28]. In this circumstance, since the structure of the original model is destroyed, one can not go back to the second order model again if it is desired in the subsequent computations.

During the recent years structure preserving MOR of second order systems received a lot of attention, see e.g., [24, 6, 1, 26] and the references therein. But all of those approaches are only for standard second order systems. As in our earlier work [28, 29, 7] on this model, also in this article our goal is to apply MOR techniques to the high dimensional model in (1) and replace it by the substantially lower-dimensional model

$$\begin{aligned} \hat{M}\ddot{\hat{z}}(t) + \hat{D}\dot{\hat{z}}(t) + \hat{K}\hat{z}(t) &= \hat{H}u(t), \\ \hat{y}(t) &= \hat{H}^T\hat{z}(t) + \hat{D}_s u(t), \end{aligned} \tag{2}$$

where $\hat{M}, \hat{D}, \hat{K} \in \mathbb{R}^{l\times l}$, $\hat{H} \in \mathbb{R}^{l\times p}$ and $l \ll n$. It is required that $\|y - \hat{y}\|$ is small and the ROM preserves necessary properties, e.g., stability, passivity and symmetry of the original model.

This paper is concerned with balancing based structure-preserving MOR of the second order index-1 system (1). The central idea of this method is to truncate the less important states from the system, which correspond to the negligible system Hankel singular values (HSVs). The system Hankel singular values are the square roots of the eigenvalues of the product of the controllability and observability Gramians [13] or equivalently the singular values of the product of the two Gramina factors [27]. A system is balanced if both the Gramians are identical and diagonal with decreasingly ordered entries which are the systems HSVs. Therefore, as a primary task in this method, by balancing the system one can easily identify which states are to be truncated.

It is known that the most expensive part in the balancing based MOR method is to solve the two Lyapunov equations determining the system Gramian factors which are the key ingredients in the derivation of the truncating projection matrices for forming the reduced order model. For a large sparse LTI system LRCF-ADI (low-rank Cholesky factor-alternating direction implicit) [5, 22] is one efficient method to compute these Gramian factors. We have already investigated this in [7] for a large sparse second order index-1 descriptor system. In contrast to [7], in this paper the LRCF-ADI method is updated by exploiting the symmetry properties of the systems, and computing real Gramian factors applying the ideas from [2]. Moreover, we use the residual factor based stopping criterion [3] to stop the LRCF-ADI algorithm. The presented algorithm to compute the low-rank Gramian factor is based on the second order index-1 descriptor system (1).

This article also investigates the model reduction of symmetric second order index-1 descriptor systems via projecting the system onto the dominant eigenspace of the system Gramian. In fact the transformations to obtain the ROM can be computed cheaply from the low-rank Gramian factors. This idea is originally found in [23, 17] for standard state space systems. Here, we show that the ROM obtained in this way

preserve the symmetry of the original system. The proposed techniques are applied to a piezo-actuated structural FEM model of a certain building block of a parallel kinematic machine tool. Numerical results illustrate the efficiency of the techniques comparing with the standard balanced truncation.

2 Model example

We investigate a part of the experimental machine tool as shown in Figure 1. It

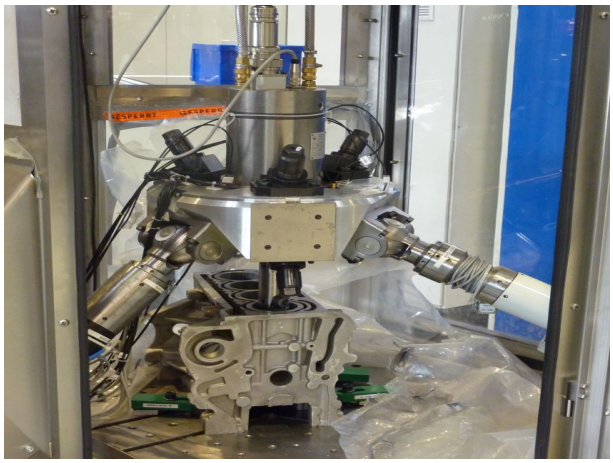


Figure 1: Piezo-actuator based mechanical system

is a complex system, where a piezo-actuator based adaptive spindle support (ASS) is mounted in a parallel kinematic machine in order to gain additional positioning freedom during machining operations (see [10, 20] for more details). The important purpose of the piezo-sensor and -actuator is to control active vibration or shunt damping so that the machine can ensure a high quality product. For analyzing the mechanical design and performance of the ASS, a mathematical model as in (1) is formed using the finite element method, where $\mathcal{M}$, $\mathcal{D}$ and $\mathcal{K}$ are the mass, damping and stiffness matrices, respectively. The time dependent state vector $z(t)$ consists of the components of mechanical displacements and $\varphi(t)$ are the electrical charges. By separating the mechanical and electrical parts, it is clear that M_1, D_1 and K_{11} are respectively, mechanical mass, damping and stiffness matrices. The matrix K is composed of mechanical (K_{11}), electrical (K_{22}) and coupling (K_{12}) terms. The general force quantities (mechanical forces and electrical charges) are chosen as the input quantities u, and the corresponding general displacements (mechanical displacements and electrical potential) are the output quantities y. The total mass matrix contains zeros at the locations of electrical potential. More precisely, the electrical potential of piezo-mechanical systems (degrees of freedom (DoF) for the electrical part) is not

associated with an inertia. The equation of motion of the mechanical system in (1) can be found in [21]. This equation results from a finite element discretization of the balance equations. For piezo-mechanical systems these are the mechanical balance of momentum (with inertia term) and the electro-static balance. From this, the electrical potential without inertia term is obtained. Thus, for the whole system (mechanical and electrical DoF) the mass matrix has rank deficiency. There are many ways to transform (1) into equivalent first order form [28]. In this paper we prefer the following representation:

$$\mathcal{E}\dot{\xi}(t) = \mathcal{A}\xi(t) + \mathcal{B}u(t), \quad y(t) = \mathcal{B}^T\xi(t) + D_a u(t). \tag{3}$$

where

$$\mathcal{E} = \begin{bmatrix} 0 & M_1 & 0 \\ M_1 & D_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathcal{A} = \begin{bmatrix} M_1 & 0 & 0 \\ 0 & -K_{11} & -K12 \\ 0 & -K_{12}^T & -K_{22} \end{bmatrix}, \mathcal{B} = \begin{bmatrix} 0 \\ B_1 \\ B_2 \end{bmatrix}, \text{ and } \xi(t) = \begin{bmatrix} \dot{z}(t) \\ z(t) \\ \varphi(t) \end{bmatrix}. \tag{4}$$

The advantage of this representation is that if the system (1) is symmetric then (3) is symmetric. Moreover, the input-output matrices are transpose of each other.

3 The BT method for second order systems and related issues

In this section we briefly review the BT method for second order LTI systems

$$\begin{aligned} M\ddot{x}(t) + D\dot{x}(t) + Kx(t) &= Hu(t), \\ y(t) &= Lx(t) + D_s u(t), \end{aligned} \tag{5}$$

where M, D and K are nonsingular, and $x(t)$ is the n dimensional state vector. Transforming (5) into first order form yields

$$\begin{aligned} \underbrace{\begin{bmatrix} 0 & M \\ M & D \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} \ddot{x}(t) \\ \dot{x}(t) \end{bmatrix}}_{\dot{\zeta}(t)} &= \underbrace{\begin{bmatrix} M & 0 \\ 0 & -K \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \dot{x}(t) \\ x(t) \end{bmatrix}}_{\zeta(t)} + \underbrace{\begin{bmatrix} 0 \\ H \end{bmatrix}}_{B} u(t) \\ y(t) &= \underbrace{\begin{bmatrix} 0 & L \end{bmatrix}}_{C} \begin{bmatrix} \dot{x}(t) \\ x(t) \end{bmatrix} + D_s u(t). \end{aligned} \tag{6}$$

The controllability Gramian $W_c \in R^{2n\times 2n}$ and the observability Gramian $W_o \in R^{2n\times 2n}$ for the system (6) are the solutions of the Lyapunov equations

$$AW_cE^T + EW_cA^T = -BB^T \quad \text{and} \quad A^TW_oE + E^TW_oA = -C^TC. \tag{7}$$

The Gramians can also be defined from a physical point of view. Defining an energy function

$$J(u) = \int_{-\infty}^{0} u^*(t)u(t)dt,$$

Golver in [13] shows that the optimal value of the minimization problem

$$\begin{aligned} &\min_{u} \ J(u) \\ \text{s. t.} \quad &E\dot{\zeta}(t) = A\zeta(t) + Bu(t), \quad \zeta(0) = \zeta_0, \end{aligned} \tag{8}$$

is

$$\zeta_0^T W_c^{-1} \zeta_0. \tag{9}$$

Which is the required minimal energy to steer the state of the system (6) from $t = -\infty$ to the state ζ_0 at time $t = 0$. Based on the optimization problem (8) the Gramians for the second order system (5) are first defined in [19]. Let us consider the following optimization problems

$$\begin{aligned} &\min_{\dot{x}_0} \min_{u} \ J(u) \\ \text{s. t.} \quad &M\ddot{x}(t) + D\dot{x}(t) + Kx(t) = Hu(t), \quad \dot{x}(0) = \dot{x}_0, \end{aligned} \tag{10}$$

and

$$\begin{aligned} &\min_{x_0} \min_{u} \ J(u) \\ \text{s. t.} \quad &M\ddot{x}(t) + D\dot{x}(t) + Kx(t) = Hu(t), \quad x(0) = x_0. \end{aligned} \tag{11}$$

Due to the structure of the system (5) the controllability Gramian can be compatibly partitioned as

$$W_c = \begin{bmatrix} P_v & P_o \\ P_o^T & P_p \end{bmatrix}.$$

The authors in [19] (see also [8]), prove that the optimal solution to the problem (10) is $\dot{x}_0 P_v^{-1} \dot{x}_0$, which is the minimal energy required to reach the given velocity $\dot{x}_0$ over all past inputs and initial values. The solution of the problem (11) is $x_0 P_p^{-1} x_0$, which is the minimal energy required to reach the given position x_0 overall past inputs and initial values. Here P_v and P_p are defined as second order controllability velocity Gramian and position Gramian, respectively. Analogously, one can interpret the observability Gramian W_o by using duality arguments. Then partitioning W_o as

$$W_o = \begin{bmatrix} Q_v & Q_o \\ Q_o^T & Q_p \end{bmatrix},$$

it can be proven that Q_v and Q_p are respectively, the observability velocity Gramian and position Gramian.

We consider R as a low rank controllability Gramian factor such that $W_c \approx RR^T$. The structure of the first order system allows us to split R as

$$R = \begin{bmatrix} R_v^T & R_p^T \end{bmatrix}^T. \tag{12}$$

Therefore, the controllability Gramian can be written as

$$W_c = \begin{bmatrix} P_v & P_o \\ P_o^T & P_p \end{bmatrix} \approx RR^T = \begin{bmatrix} R_v \\ R_p \end{bmatrix} \begin{bmatrix} R_v^T & R_p^T \end{bmatrix} = \begin{bmatrix} R_v R_v^T & R_v R_p^T \\ R_p R_v^T & R_p R_p^T \end{bmatrix}.$$

Hence we have

$$P_v \approx R_v R_v^T \qquad \text{and} \qquad P_p \approx R_p R_p^T.$$

Similarly, considering $W_o \approx LL^T$ we have

$$Q_v \approx L_v L_v^T \qquad \text{and} \qquad Q_p \approx L_p L_p^T,$$

where $L = \begin{bmatrix} L_v^T & L_p^T \end{bmatrix}^T$. Apparently, R_v and R_p are obtained from the first n rows and the lower n rows of R, respectively. Analogously, L_v and L_p can be obtained from the first n rows and the lower n rows of the low-rank observability Gramian factor L. Once we have R_α and L_β where $\alpha \in \{v, p\}$ and $\beta \in \{v, p\}$, the balancing transformation can be formed using the SVD

$$R_\alpha^T M L_\beta = U_{\alpha\beta} \Sigma_{\alpha\beta} V_{\alpha\beta}^T = \begin{bmatrix} U_{\alpha\beta,1} & U_{\alpha\beta,2} \end{bmatrix} \begin{bmatrix} \Sigma_{\alpha\beta,1} & \\ & \Sigma_{\alpha\beta,2} \end{bmatrix} \begin{bmatrix} V_{\alpha\beta,1}^T \\ V_{\alpha\beta,2}^T \end{bmatrix}, \tag{13}$$

and defining

$$T_L := L_\beta U_{\alpha\beta,1} \Sigma_{\alpha\beta,1}^{-\frac{1}{2}}, \quad T_R := R_\alpha V_{\alpha\beta,1} \Sigma_{\alpha\beta,1}^{-\frac{1}{2}}. \tag{14}$$

Here $U_{\alpha\beta,1}$ and $V_{\alpha\beta,1}$ are composed of the leading k columns of $U_{\alpha\beta}$ and $V_{\alpha\beta}$, respectively. And $\Sigma_{\alpha\beta,1}$ is the first $k \times k$ block of the matrix $\Sigma_{\alpha\beta}$. Now, the reduced order model as in (2) can be formed by constructing the matrices of reduced dimensions:

$$\hat{M} = T_L^T M_1 T_R, \ \hat{D} = T_L^T D_1 T_R, \ \hat{K} = T_L^T K T_R, \ \hat{H} = T_L^T H, \text{ and } \hat{L} = L T_R. \tag{15}$$

When $\alpha = \beta = v$, the balancing technique by the above procedure is called velocity-velocity (VV) balancing. Likewise position-position (PP) balancing is obtained if $\alpha = \beta = p$, velocity-position (VP) balancing is obtained if $\alpha = v, \beta = p$ and position-velocity (PV) balancing is obtained if $\alpha = p, \beta = v$. See, e.g., [24] for more details.

The fundamental draw back of the balancing based model reduction is to compute the two Gramian factors by solving two respective Lyapunov equations. Among several approaches the low-rank Cholesky factor alternating direction implicit (LRCF-ADI) method has been developed that allow to exploit the fact, that often all coefficient matrices are sparse and the number of inputs and outputs are very small compared to the number of DoFs. We refer the reader to [5] and the references therein for details

Algorithm 1: GLRCF-ADI for solving $FX\check{E}^T + \check{E}XF^T = -NN^T$.

Input : $F,\ \check{E},\ N$, shift parameters $\{\mu_i\}_{i=1}^{J} \subset \mathbb{C}_-$.
Output: $Z = Z_i$ such that $X \approx ZZ^T$.

$W_0 = N, \quad Z_0 = [\,], \quad i = 1.$
while $\|W_{i-1}^T W_{i-1}\| \geq tol$ *or* $i \leq i_{max}$ **do**
  $V_i = (F + \mu_i \check{E})^{-1} W_{i-1}$ for V_i.
  if Im $(\mu_i) = 0$ **then**
    $Z_i = [Z_{i-1},\ \sqrt{-2\mu_i} V_i]$.
    $W_i = W_{i-1} - 2\,\mathrm{Re}\,(\mu_i)\,\check{E} V_i$.
  else
    $\gamma_i = 2\sqrt{-\mathrm{Re}\,(\alpha_i)}, \quad \delta_i = \frac{\mathrm{Re}\,(\alpha_i)}{\mathrm{Im}\,(\alpha_i)}$.
    $Z_{i+1} = [Z_{i-1},\ \ \gamma_j\,(\mathrm{Re}\,(V_i) + \delta_j\,\mathrm{Im}\,(V_i))\,,\ \ \gamma_i\sqrt{(\delta_i^2+1)}\cdot \mathrm{Im}\,(V_i)]$.
    $W_{i+1} = W_{i-1} + \gamma_i^2 \check{E}\,(\mathrm{Re}\,(V_i) + \delta_i\,\mathrm{Im}\,(V_i))$.
    $i = i + 1$
  $i = i + 1$

on the LRCF-ADI approach. Recently this prominent method has been updated by exploiting the ideas of computing real low-rank Gramian factors [2] and the low-rank residual based stopping criterion. For convenience the updated version of the LRCF-ADI (GLRCF-ADI) is summarized in Algorithm 1.

This algorithm either successively computes $Z = R$ for $(\check{E}, F, N) = (E, A, B)$ or $Z = L$ for $(\check{E}, F, N) = (E^T, A^T, C^T)$. In this algorithm $\{\mu_i\}_{i=1}^{J}$ are called the optimal ADI shift parameters or simply shift parameters [22]. A set of optimal shift parameters is necessary for fast convergence of the algorithm. Although several strategies are available in the literature (see for example [4] and the references therein), in this article we restrict ourselves to the *heuristic* approach introduced in [22] and an adaptive choice following [4]. The algorithm can be stopped by using either a user defined tolerance *tol* or considering a maximum number of iteration steps $i_{\max}$.

4 Model reduction of second order index-1 systems

This section discusses the balancing based model reduction methods for the second order index-1 descriptor system (1). The first subsection shows, how one can apply balanced truncation to the system (1). While in the next subsection we discuss that the balancing based model reduction can be performed via projecting the system onto the dominant eigenspace of the systems Gramian.

type	SVD	left proj. T_L	right proj. T_R
VV	$Z_v^T M_1 Z_v = U_{vv}\Sigma_{vv}U_{vv}^T$	$Z_v U_{vv,1}\Sigma_{vv,1}^{-\frac{1}{2}}$	$Z_v U_{vv,1}\Sigma_{vv,1}^{-\frac{1}{2}}$
PP	$Z_p^T M_1 Z_p = U_{pp}\Sigma_{pp}U_{pp}^T$	$Z_p U_{pp,1}\Sigma_{pp,1}^{-\frac{1}{2}}$	$Z_p U_{pp,1}\Sigma_{pp,1}^{-\frac{1}{2}}$
VP	$Z_v^T M_1 Z_p = U_{vp}\Sigma_{vp}V_{vp}^T$	$Z_p U_{vp,1}\Sigma_{vp,1}^{-\frac{1}{2}}$	$Z_v V_{vp,1}\Sigma_{vp,1}^{-\frac{1}{2}}$
PV	$Z_p^T M_1 Z_v = U_{pv}\Sigma_{pv}V_{pv}^T$	$Z_v U_{pv,1}\Sigma_{pv,1}^{-\frac{1}{2}}$	$Z_p V_{pv,1}\Sigma_{pv,1}^{-\frac{1}{2}}$

Table 1: Balancing transformations for the second order index-1 descriptor systems

4.1 Balanced truncation

First of all transform the second order index-1 descriptor system (1) into a standard system (5) where

$$\begin{aligned} &M = M_1, \quad D = D_1, \quad K = K_{11} - K_{12}K_{22}^{-1}K_{12}^T, \\ &H = B_1 - K_{12}K_{22}^{-1}B_2, \quad L = H^T \quad \text{and} \quad D_s = D_a + C_2K_{22}^{-1}B_2. \end{aligned} \tag{16}$$

The first order representation of this standard second order model is obtained as in (6). Since the first order form is symmetric ($A^T = A$, $E^T = E$) and the input-output matrices are transposes of each other ($C = B^T$), the controllability Gramian and the observability Gramian coincide, i.e., $W_c = W_o = W$ and only one Lyapunov equation

$$AWE + EWA = -BB^T \tag{17}$$

needs to be solved. Therefore we can consider

$$\begin{bmatrix} R_v^T & R_p^T \end{bmatrix}^T = \begin{bmatrix} L_v^T & L_p^T \end{bmatrix}^T = \begin{bmatrix} Z_v^T & Z_p^T \end{bmatrix}^T. \tag{18}$$

Once we have Z_v and Z_p by solving (17), following (13-14) we compute four types of balancing transformations as shown in the Table 1. Now using these projectors we obtain four types of reduced order model as in (2). In each case the reduced dimensional matrices are constructed as:

$$\begin{aligned} &\hat{M} = T_L^T M_{11} T_R, \quad \hat{D} = T_L^T D_{11} T_R, \\ &\hat{K} = T_L^T K_{11} T_R - T_L^T K_{12} K_{22}^{-1} K_{21}^T T_R, \\ &\hat{H} = T_L^T B_1 - T_L^T K_{12} K_{22}^{-1} B_2, \\ &\hat{D}_s = D_a + C_2 K_{22}^{-1} B_2. \end{aligned} \tag{19}$$

4.2 Projecting onto dominant eigen-spaces of the Gramian (PDEG)

The model reduction technique via projecting the system onto the dominant eigenspace of the systems Gramian is first introduced in [23]. However, there the proposed algorithm is for standard state space systems. Here we extend the idea for the specially

Algorithm 2: BT-MOR for second order index-1 system (SR method).

Input : M_1, D_1, K_{11}, K_{12}, K_{22}, B_1, B_2, and D_S from (1).
Output: $\hat{M}$, $\hat{D}$, $\hat{K}$, $\hat{H}$ and $\hat{D}_a$ as in (2).

1 Solve Lyapunov equation (17) to compute Z_v and Z_p.
2 Compute four types of transformations following Table 1
3 Construct $\hat{M}$, $\hat{D}$, $\hat{K}$, $\hat{H}$, and $\hat{D}_s$ following (19)

structured second order index-1 systems. We already have defined the controllability velocity Gramian P_v and the observability velocity Gramian Q_v in Section 3. For our special model, they are equal and we assume that $P_v = Q_v = \mathcal{G}_v$. Since $\mathcal{G}_v$ is symmetric positive definite (SPD), it has a symmetric decomposition, i.e.,

$$\mathcal{G}_v = \mathcal{R}_v \mathcal{R}_v^T. \tag{20}$$

The *thin* SVD of R_v is

$$\mathcal{R}_v = U_v \Sigma_v V_v^T, \tag{21}$$

where the diagonal matrix Σ_v consists of the decreasingly ordered singular values σ_{v_i}, $i = 1, 2, \ldots, n_1$ of $\mathcal{R}_v$. Using this SVD we obviously have

$$\mathcal{G}_v = (U_v \Sigma_v V_v^T)(V_v \Sigma_v U_v^T) = U_v \Sigma_v^2 U_v^T. \tag{22}$$

This is also an eigenvalue decomposition, Σ_v^2 is a diagonal matrix whose entries are the decreasingly ordered eigenvalues of $\mathcal{G}_v$ and U_v is the orthogonal matrix consisting of the eigenvectors corresponding to the eigenvalues. We observe that U_v is the left singular vector matrix of $\mathcal{R}_v$. Hence U_v is obtained by the SVD of $\mathcal{R}_v$. Now identifying the k largest eigenvalues of $\mathcal{G}_v$, i.e., the k largest singular values of $\mathcal{R}_v$, construct

$$U_k = \begin{bmatrix} u_1, u_2, \ldots, u_k \end{bmatrix}, \tag{23}$$

where u_i, $i = 1, 2, \ldots, k$ are the eigenvectors corresponding to the eigenvalues σ_i^2. Then we construct the k dimensional reduced order model (2), by forming matrices as in (19) where $T_L = T_R = U_k$. Again, if we consider Z_v as a low-rank Gramian factor of the velocity Gramian such that $\mathcal{G}_v \approx Z_v Z_v^T$, then we can compute U_k in (23) identifying the k largest left singular vectors of the SVD of Z_v.

The above procedure that constructs a k dimensional ROM (2) via projecting the system onto the dominant eigen-spaces of the velocity Gramian $\mathcal{G}_v$ is summarized in Algorithm 3. However, Algorithm 3 can also be used to obtain a k dimensional ROM via projecting the system onto the eigen-space of the position Gramian $\mathcal{G}_p$, where $P_p = Q_p = \mathcal{G}_p$. In that case we will compute Z_p, the position Gramian factor such that $\mathcal{G}_p = Z_p Z_p^T$ in Step 2. Note that the pre-assigned order k of the reduced order model should satisfies the inequality

$$k \leq \dim(Z_v), \quad \text{or} \quad k \leq \dim(Z_p).$$

Algorithm 3: BT-MOR for second order index-1 system (PDEG method)

Input : M_1, D_1, K_{11}, K_{12}, K_{22}, B_1, B_2 from (1) and k (dimension of ROM).
Output: $\hat{M}$, $\hat{D}$, $\hat{K}$, $\hat{H}$ and $\hat{D}_s$ as in (2).

1 Compute Z_v by solving Lyapunov equation;
2 Construct U_k as in (23) using SVD of Z_v;
3 Following (19) where T_L and T_R are replaced by U_k, form reduced dimensional matrices $\hat{M}$, $\hat{D}$, $\hat{K}$, $\hat{H}$ and $\hat{D}_s$

The transformation U_k is called *contra-gradient transformation* [16], since using this transformation we can show that

$$\begin{aligned} U_k^T \mathcal{G}_v U_k &= U_k^T U_v \Sigma_v^2 U_v^T U_k \\ &= U_k^T \begin{bmatrix} U_k & U_{n_1-k} \end{bmatrix} \begin{bmatrix} \Sigma_k^2 & 0 \\ 0 & \Sigma_{n_1-k}^2 \end{bmatrix} \begin{bmatrix} U_k^T \\ U_{n_1-k}^T \end{bmatrix} U_k \\ &= \begin{bmatrix} I_k & 0 \end{bmatrix} \begin{bmatrix} \Sigma_k^2 & 0 \\ 0 & \Sigma_{n_1-k}^2 \end{bmatrix} \begin{bmatrix} I_k \\ 0 \end{bmatrix} \\ &= \Sigma_k^2, \end{aligned}$$

i.e., the Gramian of the reduced model is diagonal. That is U_k is the balancing transformation. It can easily be shown that $\hat{M}$, $\hat{D}$ and $\hat{K}$ are all symmetric. Moreover, the ROM preserves the definiteness of the original system. Therefore, the stability is also preserved.

5 Compute the low-rank Gramian factors for the second order index-1 systems

This section concentrates on the efficient computation of Z_v and Z_p as defined above for the second order index-1 DAEs (1) by solving the Lyapunov equation (17). In contrast to our previous work [29] we want to apply Algorithm 1 with all its efficiency improving features. In the following we discuss some computational strategies.

We know that the most expensive part in the LRCF-ADI iteration is to solve a linear system in each iteration step. The linear system can be solved directly or iteratively. In either cases, for our problem, avoiding the Schur complement formulation, i.e., $(K_{11} - K_{12}K_{22}^{-1}K_{12}^T)$ and exploiting the second order block structure we can accelerate the computation. In the following we discuss these issues.

When we solve the Lyapunov equation (17) by applying Algorithm 1, in the $i-$th step (see Step 3 in Algorithm 1), we need to compute $V_i = (A + \mu_i E)^{-1} W_{i-1}$, where E and A are defined in (6). Let us consider

$$(A + \mu_i E)^{-1} W_{i-1} = \chi,$$

or

$$(A + \mu_i E)\chi = W_{i-1}. \tag{24}$$

Which is equivalent to

$$\begin{bmatrix} M & \mu_i M \\ \mu_i M & \mu_i D - K \end{bmatrix} \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} = \begin{bmatrix} W_{i-1}^{(1)} \\ W_{i-1}^{(2)} \end{bmatrix}. \tag{25}$$

Now, inserting M, D and K from (16), linear system (25) implies

$$\begin{bmatrix} M_1 & \mu_i M_1 \\ \mu_i M_1 & \mu_i D_1 - (K_{11} - K_{12}K_{22}^{-1}K_{12}^T) \end{bmatrix} \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} = \begin{bmatrix} W_{i-1}^{(1)} \\ W_{i-1}^{(2)} \end{bmatrix}. \tag{26}$$

It can easily be shown that reversing the Schur complement instead of solving the linear system (26) we can solve the linear system

$$\begin{bmatrix} M_1 & \mu_i M_1 & 0 \\ \mu_i M_1 & \mu_i D_1 - K_{11} & -K_{12} \\ 0 & -K_{12}^T & -K_{22} \end{bmatrix} \begin{bmatrix} \chi_1 \\ \chi_2 \\ \Gamma \end{bmatrix} = \begin{bmatrix} W_{i-1}^{(1)} \\ W_{i-1}^{(2)} \\ 0 \end{bmatrix}, \tag{27}$$

for $\left[\chi_1^T, \chi_2^T\right]^T$. Although the dimension of the matrices in (27) is higher than that of (26), it is sparse and therefore it can be treated using a sparse direct solver [9, Ch. 5], or any suitable iterative solver [25]. The computation can be accelerated further by splitting the linear system (27) as follows:
A simple algebraic manipulation on (27), again leads us first to solve the linear system

$$\begin{bmatrix} \mu_i^2 M_1 - \mu_i D_1 + K_{11} & K_{12} \\ K_{12}^T & K_{22} \end{bmatrix} \begin{bmatrix} \chi_2 \\ \Gamma \end{bmatrix} = \begin{bmatrix} \mu_i W_{i-1}^{(1)} - W_{i-1}^{(2)} \\ 0 \end{bmatrix} \tag{28}$$

for χ_2, then to compute $\chi_1 = M_1^{-1} W_{i-1}^{(1)} - \mu_i \chi_2$. Here $W_{i-1}^{(1)}$ and $W_{i-1}^{(2)}$ are already computed from the previous step (from the ADI residual) by following the expressions

$$\begin{aligned} W_i^{(1)} &= W_{i-1}^{(1)} - \gamma \mu_i M_1 \chi_2, \\ W_i^{(2)} &= W_{i-1}^{(2)} - \gamma \mu_i (M_1 \chi_1 + D_1 \chi_2), \end{aligned} \tag{29}$$

where $\gamma = 2\,\mathrm{Re}\,(\mu_i)$. This relation can easily be obtained by splitting W_i in Step 6 in Algorithm 1 as $\left[{W_i^{(1)}}^T \quad {W_i^{(1)}}^T\right]^T$. In case, the two consecutive shift parameters are complex conjugates of each other, i.e., $\{\mu_i, \mu_{i+1} := \overline{\mu_i}\}$, (29) should be replaced by

$$\begin{aligned} W_{i+1}^{(1)} &= W_{i-1}^{(1)} - 2\gamma M_1 \chi_2, \\ W_{i+1}^{(2)} &= W_{i-1}^{(2)} - 2\gamma (M_1 \chi_1 + D_1 \chi_2). \end{aligned}$$

Algorithm 4: SOGS-LRCF-ADI for the second order index-1 systems

Input : $M_1, D_1, K_{11}, K_{12}, K_{22}, B_1, B_2$ and shift parameters $\{\mu_i\}_{i=1}^{J}$.

Output: Z, Z_v and Z_p, where $Z = Z_i = \begin{bmatrix} Z_v^T & Z_p^T \end{bmatrix}^T$ with $W \approx ZZ^T$.

Set $Z_0 = []$, $i = 1$, $W_0^1 = 0$ and $W_0^2 = B_1 - K_{12}K_{22}^{-1}B_2$.

while $\|{W_{i-1}^{(1)}}^T W_{i-1}^{(1)} + {W_{i-1}^{(2)}}^T W_{i-1}^{(2)}\| \geq tol$ *or* $i \leq i_{max}$ **do**

  Solve $\begin{bmatrix} \mu_i^2 M_1 - \mu_i D_1 + K_{11} & K_{12} \\ K_{12}^T & K_{22} \end{bmatrix} \begin{bmatrix} \chi_2 \\ \Gamma \end{bmatrix} = \begin{bmatrix} \mu_i W_{i-1}^{(1)} - W_{i-1}^{(2)} \\ 0 \end{bmatrix}$ for χ_2.

  Compute $V_i = \begin{bmatrix} (M_1^{-1} W_{i-1}^{(1)} - \mu_i \chi_2)^T & \chi_2^T \end{bmatrix}^T$.

  if $\mathrm{Im}(\mu_i) = 0$ **then**

    $Z_i = \begin{bmatrix} Z_{i-1} & \sqrt{\gamma}\,\mathrm{Re}\,(V_i) \end{bmatrix}$, where $\gamma = -2\,\mathrm{Re}\,(\mu_i)$,

    $W_i^{(1)} = W_{i-1}^{(1)} + \gamma M_1 \chi_2, \quad W_i^{(2)} = W_{i-1}^{(2)} + \gamma(M_1\chi_1 + D_1\chi_2)$.

  else

    $\delta = \frac{\mathrm{Re}\,(\mu_i)}{\mathrm{Im}(\mu_i)}, \quad V_{i+1} = \overline{V}_i + 2\delta\,\mathrm{Im}(V_i)$.

    Update low-rank solution factor

    $Z_{i+1} = \begin{bmatrix} Z_{i-1} & \sqrt{2\gamma}(\mathrm{Re}\,(V_i) + \delta\,\mathrm{Im}(V_i)) & \sqrt{2\gamma}\sqrt{(\delta^2+1)}.\,\mathrm{Im}(V_i) \end{bmatrix}$.

    Compute $\chi_1 = \mathrm{Re}\,(\chi_1) + \delta\,\mathrm{Im}(\chi_1)$, $\chi_2 = \mathrm{Re}\,(\chi_2) + \delta\,\mathrm{Im}(\chi_2)$,

    and $W_{i+1}^{(1)} = W_{i-1}^{(1)} + 2\gamma M_1 \chi_2, \quad W_{i+1}^{(2)} = W_{i-1}^{(2)} + 2\gamma(M_1\chi_1 + D_1\chi_2)$.

    $i = i + 1$

  $i = i + 1$

Thus, in the i-th step V_i can be computed by

$$V_i = \chi = \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} = \begin{bmatrix} M_1^{-1} W_{i-1}^{(1)} - \mu_i \chi_2 \\ \chi_2 \end{bmatrix}$$

The whole procedure is presented in Algorithm 4. Note that in this algorithm to compute the exact residual our initial guess is $W_0 = B = \begin{bmatrix} 0 \\ H \end{bmatrix}$, which can be again spitted as $W_0^1 = 0$ and $W_0^2 = H = B_1 - K_{12}K_{22}^{-1}B_2$.

Shift parameters selection. It is known that for fast convergence of Algorithm 4 proper ADI shift parameters selection is crucial. Several approaches are proposed in the literature to compute the shift parameters. See, e.g., [4] for an overview of different shift selection approaches. For a large scale dynamical system an often used ADI shift selection technique is *Penzl's heuristic* procedure [22]. In [4] the authors propose an automatic shift generation technique that works during the execution of the algorithm, rather than before. There, the technique is called *adaptive procedure* and numerical experiments shows that the approach performs very well for a first order index-1 power system model [12]. Here, we investigate both the techniques and propose a modification to the adaptive shift selection procedure.

The heuristic procedure, to select some suboptimal ADI parameters for the Algorithm 4 has been discussed in our previous work (see e.g., [28, Algorithm 4.4], [7]). For a second shift selection technique we recall [4]. There, the shifts are initialized by the eigenvalues of the pencil $\lambda E - A$ projected to the span of B. Then, whenever all shifts in the set have been used, the pencil is projected to the span of the current V_i and the eigenvalues are used as the new set of shifts. Here, we use the same initialization. For the update step however we extend the subspace to all the V_i generated with the previous set of shifts. Let us assume that this (orthogonalized) extended subspace is U. Now from the eigenvalues of $\lambda U^T E U - U^T A U$, select some desired number of shifts by solving the so called ADI min-max problem like in the heuristic procedure. Repeat this approach while the algorithm has not converged within the given tolerance. Note that our system is dissipative, i.e., all the eigenvalues of $\lambda(E + E^T) - (A + A^T)$ lie in the left complex plane. Therefore, Bendixon's theorem [18] ensures that all the eigenvalues of the projected pencil are stable and thus are admissible shifts.

6 Numerical Results

In this section we illustrate numerical results to asses the accuracy and efficiency of our proposed techniques. The techniques are applied to a set of data for the finite element discretization of an adaptive spindle support (ASS) [15]. The dimension of the original model is $n = 290\,137$, which consists of $n_1 = 282\,699$ differential equations and $n_2 = 7\,438$ algebraic equations.

In this paper all the results have been obtained using MATLAB 7.11.0 (R2010b) on a board with 4 INTEL XEON E7-8837 CPUs with a 2.67-GHz clock speed, 8 Cores each and 1TB of total RAM.

To execute both Algorithms 2 and 3 we compute low-rank Gramian factor Z using Algorithm 4. To implement this algorithm we use both heuristic and adaptive shift parameters as mentioned in Section 5. First we consider 40 heuristic shifts out of 60 large and 50 small magnitude approximate eigenvalues (see [7] for details on the computation of heuristic ADI shift parameters for the ASS model). The algorithm is stopped by the maximum number of iteration steps, i.e., $i_{\max} = 400$. Next we apply the adaptive shift computation approach to compute the Z. In this case, $i_{\max} = 400$ iteration steps are taken. As we can see in Table 2 the performance of the adaptive shifts better than that of the heuristic shifts in terms of the the normalized residual norm. Note that before forming Z_v and Z_p by taking the upper n_1 rows and lower n_1 rows of Z we apply the column compression on the low-rank Gramian factor Z (see, e.g., [14] for the column compression technique).

6.1 Model reduction via balanced truncation

First we apply Algorithm 2 to the ASS model considering the truncation tolerance 10^{-5}. In this case different second order balancing approaches compute different dimensional reduced systems as shown in Table 3. The comparisons of the full and different dimensional reduced systems are shown in Figure 2 on the wide range 10^1

no. of iteration	normalized residual norm	
	heuristic shifts	adaptive shifts
100	9.88×10^{-1}	1.85×10^{-2}
200	9.99×10^{-1}	8.85×10^{-3}
300	9.78×10^{-1}	5.04×10^{-3}
400	9.69×10^{-1}	3.99×10^{-3}

Table 2: The performances of the heuristic and adaptive shifts in the Algorithm 4 for the ASS model.

label	ROM dimension
velocity-velocity (VV)	298
position-position (PP)	69
velocity-position (VP)	90
position-velocity (PV)	90

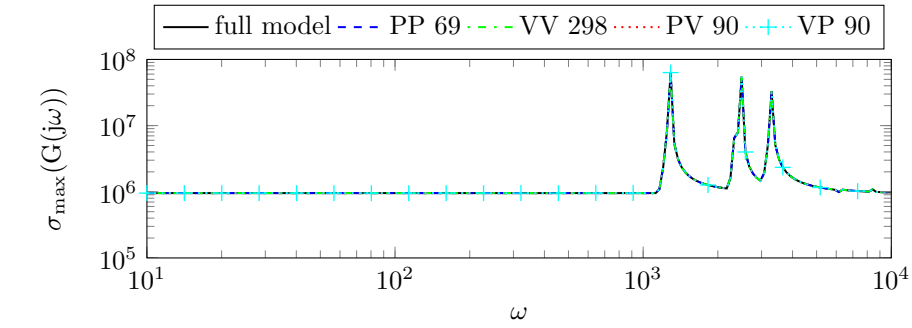
Table 3: Balancing on different levels and to different dimensional ROMs.

to 10^4. Figure 2a shows the frequency responses of full and reduced systems with good matching. The absolute error and the relative error of the frequency responses of full and reduced systems are exhibited in Figure 2b and Figure 2c respectively. As we can see in Figure 2c, the relative errors for all reduced systems are far below to the truncation tolerance (10^{-5}). We further compute the $40, 30, 20$ and 10 dimensional reduced order models using the same algorithm via balancing the system on the position-position level. In this case the frequency responses of the reduced systems also resemble the graph in Figure 2a. Figure 3 depicts the relative errors between the full and different dimensional reduced order models. Here we observe that the lower the dimension of the reduced models the higher the relative error. But in every case, even a very low dimensional model, e.g., the model of dimension 10, preserves the important feature of the original model.

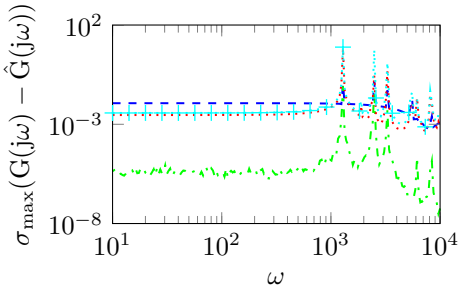
Figure 4 discusses some particularly interesting SISO relations for full and different dimensional reduced order models. Since in the SISO case we know that the transfer function matrix is just a scalar rational function, here we have computed the absolute values of the transfer function in different frequencies. The relative error between the original and reduced order models of the respective SISO relation are also shown in the same figure.

6.2 Model reduction via PDEG

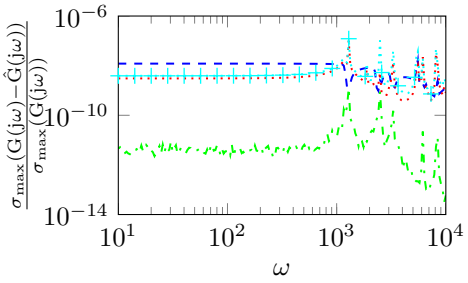
Algorithm 3 is applied again on the ASS model to obtain the reduced systems via projecting the system onto the dominant eigenspace of the Gramian. To execute this algorithm, the computed velocity and position Gramian factors are the same as mentioned earlier. By predefining the dimension of the ROM we compute 40, 30, 20 and 10 dimensional models by projecting the system onto the dominant eigenspace of



(a) Sigma plot



(b) Absolute error



(c) Relative error

Figure 2: Comparison of original and different dimensional reduced systems (dimensions indicated in the legend) computed by Algorithm 2.

the velocity Gramian. In both cases frequency responses of the original and reduced systems are seemed to be the same as in Figure 4. Figures 6 and 5, respectively, show the relative error between the original and different dimensional reduced models when we project the systems onto the dominant eigenspace of the velocity Gramian (VG) and position Gramian (PG). We observe that the constructed reduced systems of the ASS model by PDEG methods are asymptotically stable which is shown in Figure 7. This figure shows that all the eigenvalues of the reduced systems lie in the left complex half plane. From this figure one can also see that the successively decreasing dimensional reduced system contains the eigenvalues approach the imaginary axis.

7 Conclusions

In this article we have shown efficient techniques for structure preserving model reduction of second order index-1 differential algebraic systems. The performance of our proposed strategies have been demonstrated for one large FEM model of an adaptive spindle support employing piezo actuators with almost 300 000 degrees of freedom,

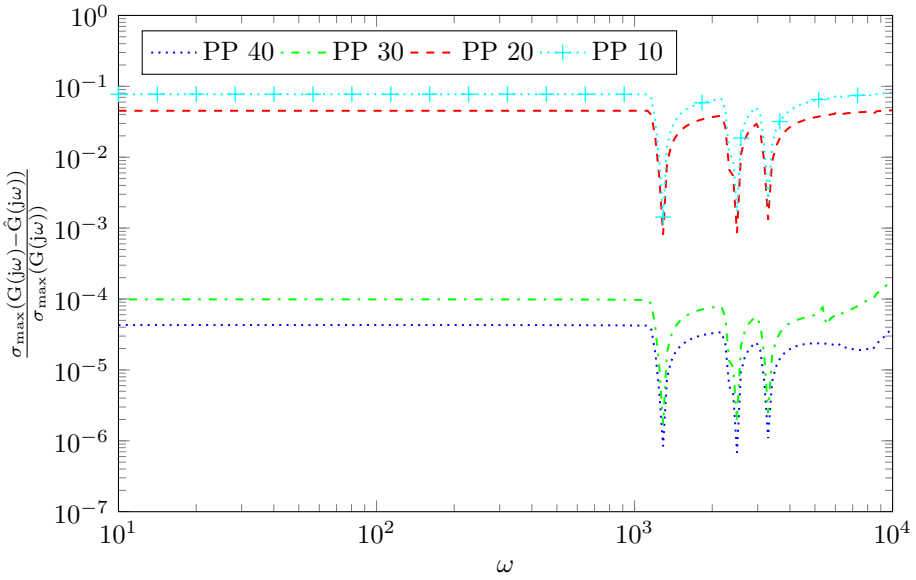


Figure 3: Relative error of full and different dimensional reduced systems (dimensions indicated in the legend) via balancing the system on the position level.

proving the applicability of our method in real world problems. In the numerical results we have seen that even 10 dimensional model preserve the main features of the original model, therefore they are expected to have a good performance in controller design. In this paper, we have discussed four types of balancing methods for second order index-1 descriptor system of the form (1) using classical balanced truncation techniques. We have also shown balancing techniques by projecting the system onto the eigenspace corresponding to the dominant eigenvalues of the Gramian. In [24], the authors claim that in general none of the existing BT techniques for second-order systems gurantees stability of the reduced systems. For the case of symmetric systems with colocated inputs and outputs stability can be guranteed by Bendixon's theorem, which is numerically depicted in Figure 7.

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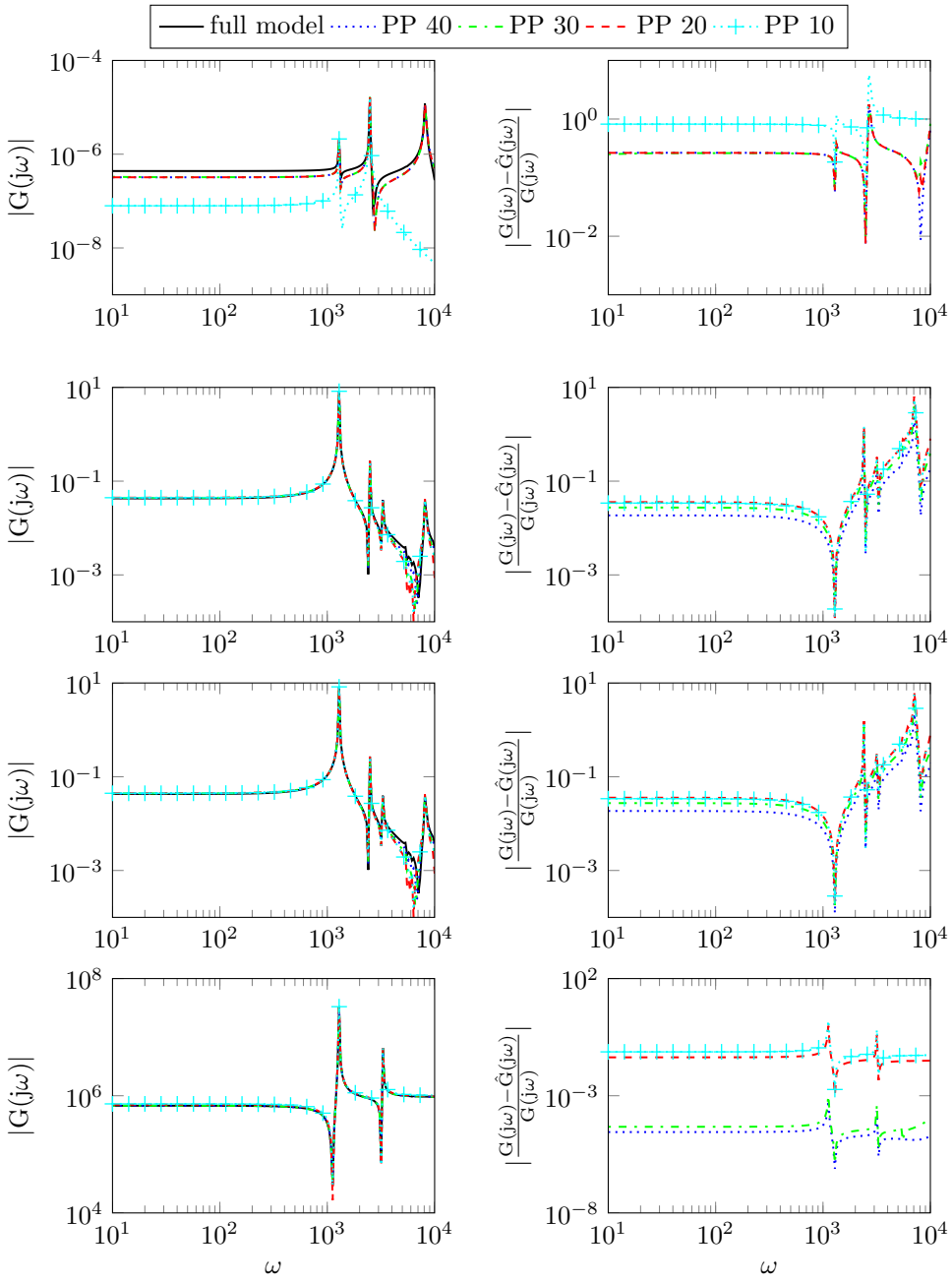

Figure 4: The rows respectively, show the 1st input to 1st output, 9th input to 1st output, 1st input to 9th and 9th input to 9th output relations (left) and the respective relative errors (right) of full and reduced systems (dimensions indicated in the legend) for position-position balancing.

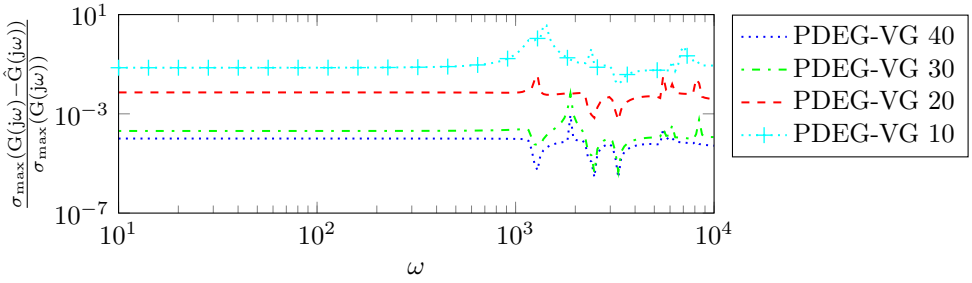

Figure 5: Relative error between full and different dimensional reduced models (dimensions indicated in the legend) via projecting onto the dominant eigensace of the position Gramian.

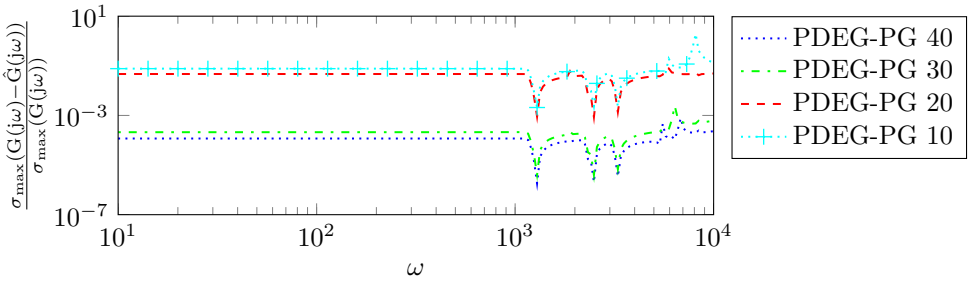

Figure 6: Relative error between full and different dimensional reduced models (dimensions indicated in the legend) via projecting onto the dominant eigenspace of the velocity Gramian.

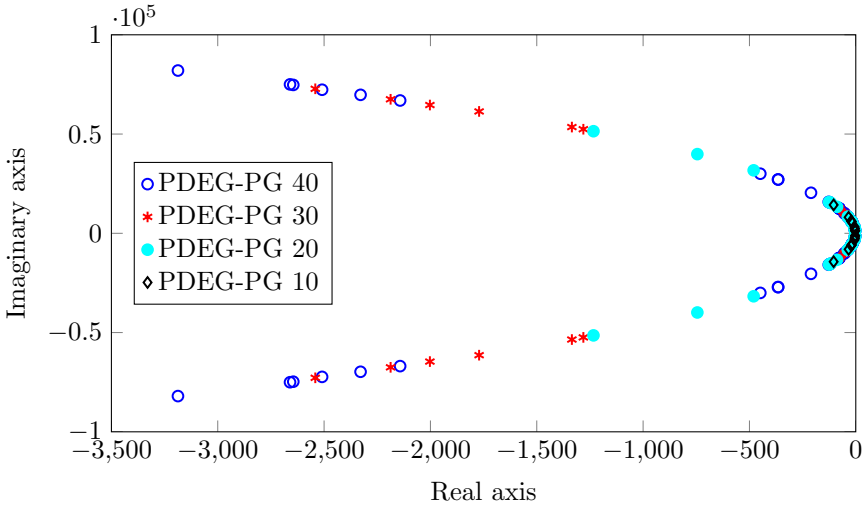

Figure 7: Eigenvalues of the ROM via projecting onto the dominant eigenspace of the position Gramian.

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