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Mian Ilyas Ahmad Ulrike Baur Peter Benner

Implicit Volterra Series Interpolation for Model Reduction of Bilinear Systems

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Abstract

We propose a new interpolatory framework for model reduction of large-scale bilinear systems. The input-output representation of a bilinear system in frequency domain involves a series of multivariate transfer functions, each representing a subsystem of the bilinear system. If a weighted sum of these multivariate transfer functions associated with a reduced bilinear system interpolates a weighted sum of the original multivariate transfer functions, we say that the reduced system satisfies a Volterra series interpolation [8]. These interpolatory conditions can also ensure the necessary conditions for $\mathcal{H}_2$-optimal model reduction [8, 5]. We observe that, by carefully selecting the weights of the series, the Volterra series interpolatory conditions are transformed to the problem of interpolating a linear system with an affine parameter dependence. Such linear parametric systems can then be reduced by some method for parametric model order reduction.

Linear systems where the affine parameter dependence is given as low-rank variation in the state matrix can be mapped into a non-parameterized multi-input multi-output linear system. This allows us to utilize the standard (non-parametric) linear IRKA [10] for the problem of parameterized/bilinear interpolation. Numerical results show that the approximations are of comparable accuracy to those obtained from the bilinear iterative rational Krylov algorithm [5]. The proposed approach, however, has the advantage that it reduces the computational costs as it involves computations associated with solving linear systems only. **Keywords.** Bilinear control systems, parametric model order reduction.

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1 Introduction

Bilinear systems are weakly nonlinear systems that have been well studied in the literature. Some applications include the modeling of electrical circuits, mechanical links, power systems [1], the heat transfer [6], fluid flow, immunology and cardiovascular regulators [11]. Often, these models typically involve a large number of state equations (the system's order) such that the simulation/control of such models is computationally expensive and prohibitive. In this paper, we discuss model order reduction of such large-scale bilinear control systems.

Consider a multi-input multi-output (MIMO) bilinear system of the form

$$\zeta : \begin{cases} \dot{x}(t) = Ax(t) + \sum_{i=1}^{m} N_i x(t) u_i(t) + Bu(t), \\ y(t) = Cx(t). \end{cases} \tag{1}$$

Here, $A, N_i \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$ are the state-space matrices and $x(t) \in \mathbb{R}^n$, $u(t) = [u_i(t), \ldots, u_m(t)]^T \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$ are the state, input and output vectors, respectively. It is assumed that ζ is bounded-input bounded-output (BIBO) stable. See [16] for the concept of stability for bilinear systems.

The problem of model reduction is to construct a reduced-order bilinear system

$$\hat{\zeta} : \begin{cases} \dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \sum_{i=1}^{m} \hat{N}_i \hat{x}(t) u_i(t) + \hat{B}u(t), \\ \hat{y}(t) = \hat{C}\hat{x}(t), \end{cases} \tag{2}$$

with $\hat{x}(t) \in \mathbb{R}^r$ and $r \ll n$ such that $\hat{y}(t)$ is close to $y(t)$ in an appropriate norm. Various techniques are available in the literature to identify $\hat{\zeta}$ for a given ζ. Analogously to the reduction of linear systems, projection is often used to construct $\hat{\zeta}$. This, in general, involves identifying suitable basis matrices V and W (the columns of each matrix span to a particular subspace), approximating $x(t)$ by $V\hat{x}(t)$ and ensuring Petrov-Galerkin conditions (with $W^TV = I$). This leads to the following reduced state-space matrices:

$$\begin{aligned} \hat{A} &= W^TAV, & \hat{N}_i &= W^TN_iV, \quad i = 1, \ldots, m, \\ \hat{B} &= W^TB, & \hat{C} &= CV. \end{aligned} \tag{3}$$

Clearly, the quality of the reduced system matrices depends on the choice of the basis matrices V and W. Various approaches for model reduction of linear systems have been extended to bilinear systems such that an efficient choice of these basis matrices can be identified. For example, it is shown in [16, 6] that an extension of the balanced truncation technique [9] to bilinear systems identifies a good choice for V and W. The approach, however, requires the solution of two generalized Lyapunov equations which is known to be computationally complex [6]. Also, interpolatory model reduction techniques have been extended to bilinear systems, see for example [12, 2, 7] and the references therein. Here, the input-output characterization (Volterra series representation) of a bilinear system, that is to be reduced, identifies an infinite set of multivariate transfer functions also known as subsystems of the bilinear system. The basis matrices V and W are than constructed such that the first k subsystems

of the reduced bilinear system are interpolating the corresponding subsystems of the original bilinear system.

The problem of $\mathcal{H}_2$-optimal model reduction for bilinear systems has also received attention after its introduction in [16]. Recently, this is linked to the problem of interpolation [5] where Wilson-based necessary conditions [15] for $\mathcal{H}_2$-optimality have been extended to bilinear systems and an iterative algorithm is proposed that updates a set of interpolation points. On convergence, the optimality conditions are satisfied. This extends the iterative rational Krylov algorithm (IRKA) [10] for linear systems to bilinear systems. The algorithm is named as bilinear iterative rational Krylov algorithm (B-IRKA) and was successfully applied to several bilinear test examples, outperforming most of the available techniques for bilinear systems. An equivalent set of necessary $\mathcal{H}_2$-optimality conditions has been derived in [8] and a truncated bilinear iterative rational Krylov algorithm (TB-IRKA) has been proposed, that reduces the computational costs of B-IRKA. The reduced bilinear system, however, only approximately satisfies the optimality conditions. This link of $\mathcal{H}_2$-optimality with interpolatory projection suggests that interpolating a weighted sum of multivariate transfer functions is more useful compared to individually interpolating each multivariate transfer function. In this paper, we propose that by carefully selecting the weights of the series, the bilinear interpolation-based necessary conditions can be ensured by using linear computations only. The proposed approach is based on rewriting the bilinear interpolatory conditions as interpolatory conditions of a linear parametric system.

In Section 2, we briefly review the bilinear interpolatory conditions. It is shown in Section 3 how these interpolatory conditions are related to the interpolatory conditions of a linear parametric system. The conditions in the MIMO case are discussed in Section 4. In Section 5 it is described how the linear parametric system can be reduced by mapping the system into a non-parametric MIMO linear system and using the standard IRKA. The efficiency of the proposed approach is underlined by numerical results in Section 6. Conclusions are given in Section 7.

2 Interpolatory Techniques for Bilinear Systems

In this section, we briefly review the Volterra series representation of a single-input single-output (SISO) bilinear system. The input-output representation for general nonlinear systems can be expressed by the Volterra series expansion of the output $y(t)$ with quantities analogously to the standard convolution operator:

$$\begin{aligned} y(t) &= \sum_{k=1}^{\infty} y_k(t_1, \ldots, t_k), \\ &= \sum_{k=1}^{\infty} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} h_k(t_1, \ldots, t_k) u(t-t_1) \cdots u(t-t_k) dt_k \ldots dt_1, \end{aligned} \tag{4}$$

where it is assumed that the input signal is one-sided, $u(t) = 0$ for $t < 0$. In addition, each of the generalized impulse responses $h_k(t_1, \ldots, t_k)$, also called k-dimensional kernels of the subsystem, is also assumed to be one-sided. In terms of the multivariable Laplace transform, the k-dimensional subsystem can be represented as

$$Y_k(s_1, \ldots, s_k) = H_k(s_1, \ldots, s_k) U(s_1) \cdots U(s_k), \tag{5}$$

where $H_k(s_1, \ldots, s_n)$ is the corresponding multivariable transfer function. The above equation follows by exploiting properties of convolution in (4), see [13] for details. If the multivariable Laplace transforms $H_k(s_1, \ldots, s_k)$ of all subsystems and the input $U(s)$ are known, then the inverse Laplace transforms can be computed to identify $y_k(t_1 \ldots, t_k)$. The output $y(t)$ becomes

$$y(t) = \sum_{k=1}^{\infty} y_k(t_1, \ldots, t_k)|_{t_1=\ldots=t_k=t} = \sum_{k=1}^{\infty} y_k(t, \ldots, t). \tag{6}$$

In case of a bilinear system ζ as given in (1) and with the additional assumption that ζ is a SISO bilinear system with $N = N_1$, the n-dimensional kernels can be written as

$$h_k(t_1, \ldots, t_n) = c\, e^{At_k} N e^{At_{k-1}} N \cdots e^{At_1} b. \tag{7}$$

The MIMO case is discussed later in Section 4. The multivariable Laplace transform of the kernel becomes

$$H_k(s_1, \ldots, s_k) = c\,(s_k I - A)^{-1} N (s_{k-1} I - A)^{-1} N \cdots (s_1 I - A)^{-1} b. \tag{8}$$

For more details on the above theory, we refer to [13]. Most of the interpolatory techniques for model reduction of bilinear systems [12, 2, 7] identify a reduced bilinear system such that the first q multivariable reduced transfer functions, $\hat{H}_k(s_1, \ldots, s_k)$, $k = 1, \ldots, q$, interpolate the corresponding original transfer functions. We call these conditions for interpolatory model reduction the 'standard interpolation'.

Definition 2.1 (Standard Interpolation)**.** *Let $S_k = \{\sigma_k^{(1)}, \ldots, \sigma_k^{(r)}\}$, $k = 1, \ldots, q$, be q different sets of interpolation points. Then we say that $\hat{H}_k(s_1, \ldots, s_k)$ achieves the* standard interpolation *if*

$$H_k(\sigma_1^{(i)}, \ldots, \sigma_k^{(i)}) = \hat{H}_k(\sigma_1^{(i)}, \ldots, \sigma_k^{(i)}), \quad k = 1, \ldots, q,\ i = 1, \ldots, r. \tag{9}$$

Based on standard interpolation, one can generate reduced-order models. However in this setting, unlike for linear systems, we can not link the issue of selecting the interpolation points to the problem of $\mathcal{H}_2$-optimal model reduction. That is, we can not identify a choice of interpolation points for which the error system associated with the r dimensional bilinear system is minimized in terms of the $\mathcal{H}_2$-norm. For bilinear systems, the $\mathcal{H}_2$-norm was introduced in [16] and, analogously to the linear case, it was expressed as

$$\|\zeta\|_{\mathcal{H}_2}^2 = cPc^T, \tag{10}$$

where $P \in \mathbb{R}^{n \times n}$ solves the generalized Lyapunov equation

$$AP + PA^T + NPN^T + bb^T = 0. \tag{11}$$

The problem of $\mathcal{H}_2$-optimality for reduced bilinear systems is also considered in [16] and necessary conditions for $\mathcal{H}_2$-optimality are derived that extend the Wilson conditions to the bilinear case:

$$\begin{aligned} Y^T X + \hat{Q}\hat{P} &= 0, & \hat{Q}\hat{N}\hat{P} + Y^T N X &= 0, \\ Y^T b + \hat{Q}\hat{b} &= 0, & \hat{c}\hat{P} - cX &= 0. \end{aligned} \tag{12}$$

Here, $\hat{P}$ is the solution of the generalized Lyapunov equation corresponding to the reduced system matrices and X, Y are solutions to the generalized Sylvester equations

$$\begin{aligned} AX + X\hat{A}^T + NX\hat{N}^T + b\hat{b}^T &= 0, \\ A^TY + Y\hat{A} + N^TY\hat{N} + c^T\hat{c} &= 0. \end{aligned} \tag{13}$$

The reduced system matrices have to satisfy the conditions in (12) for $\mathcal{H}_2$-optimality. To ensure that a projection technique satisfies these conditions, we assume that there exist X and Y such that (13) holds and

$$V = XF, \quad W = YG, \tag{14}$$

with $F, G \in \mathbb{R}^{r\times r}$. Analogously to the linear case [14], one can view (3), (13) and (14) as two coupled equations, $(\hat{A}, \hat{N}, \hat{b}, \hat{c}) = \mathcal{F}_1(X, Y, F, G)$ and $(X, Y, F, G) = \mathcal{F}_2(\hat{A}, \hat{N}, \hat{b}, \hat{c})$, for which we have a fixed point $(\hat{A}, \hat{N}, \hat{b}, \hat{c}) = \mathcal{F}_1(\mathcal{F}_2(\hat{A}, \hat{N}, \hat{b}, \hat{c}))$ at every stationary point. This clearly suggests an iterative procedure as recently introduced in [5]. The algorithm on convergence satisfies the necessary conditions given in (12). An equivalent set of interpolation-based necessary conditions for $\mathcal{H}_2$-optimality is also derived in [5]. To ensure that the reduced system satisfies these interpolatory conditions, the authors in [5] developed the B-IRKA algorithm which successfully extends the linear IRKA [10] to bilinear systems. Unlike standard interpolation (9), the B-IRKA algorithm amounts to a different framework for interpolatory model reduction. This interpolatory framework is then developed in [8] by observing that the error system in terms of the $\mathcal{H}_2$-norm can be written as

$$\begin{aligned} &\|\zeta - \hat{\zeta}\|^2_{\mathcal{H}_2} = \\ &\textstyle\sum_{k=1}^{\infty}\sum_{l_1=1}^{n}\cdots\sum_{l_k=1}^{n}\phi_{l_1,\ldots,l_k}\Big(H_k(-\lambda_{l_1},\ldots,-\lambda_{l_k}) - \hat{H}_k(-\lambda_{l_1},\ldots,-\lambda_{l_k})\Big) \\ &\textstyle+\sum_{k=1}^{\infty}\sum_{l_1=1}^{r}\cdots\sum_{l_k=1}^{r}\hat{\phi}_{l_1,\ldots,l_k}\Big(H_k(-\hat{\lambda}_{l_1},\ldots,-\hat{\lambda}_{l_k}) - \hat{H}_k(-\hat{\lambda}_{l_1},\ldots,-\hat{\lambda}_{l_k})\Big), \end{aligned} \tag{15}$$

where $\phi_{l_1,\ldots,l_k}$ are the residues of $H_k(s_1,\ldots,s_k)$ that are defined analogously to the single variable case. $\lambda_1,\ldots,\lambda_n \in \mathbb{C}$ represent the simple zeros of $H_k(s_1,\ldots,s_k)$. This means that the $\mathcal{H}_2$-norm of the error system is due to the mismatch of the weighted sums of the multivariate transfer functions evaluated at different combinations of the eigenvalues. This suggested that, analogously to the linear case, one should minimize the second term in (15). This gives rise to an interpolatory model reduction technique that aims to capture the response of the underlying Volterra series with respect to a predefined set of interpolation points in r different combinations. We call this framework for interpolatory model reduction 'Volterra series interpolation'.

Definition 2.2 (Volterra Series Interpolation)**.** *Let $S = \{\sigma^{(1)},\ldots,\sigma^{(r)}\}$ be a set of interpolation points. Then we say that a reduced bilinear system $\hat{\zeta}$ achieves the* Volterra series interpolation *if it interpolates the weighted series*

$$\begin{aligned} &\textstyle\sum_{k=1}^{\infty}\sum_{l_1=1}^{r}\cdots\sum_{l_{k-1}=1}^{r}\eta_{l_1,\ldots,l_{k-1},j}H_k(\sigma^{(l_1)},\ldots,\sigma^{(l_{k-1})},\sigma^{(j)}) = \\ &\textstyle\sum_{k=1}^{\infty}\sum_{l_1=1}^{r}\cdots\sum_{l_{k-1}=1}^{r}\eta_{l_1,\ldots,l_{k-1},j}\hat{H}_k(\sigma^{(l_1)},\ldots,\sigma^{(l_{k-1})},\sigma^{(j)}), \end{aligned} \tag{16}$$

where $j = 1, \ldots, r$ *and* $\eta_{l_1,\ldots,l_{k-1},j}$ *are weights defined in terms of the elements of a matrix* $U \in \mathbb{R}^{r\times r}$ *such that* $\eta_{l_1,\ldots,l_{k-1},j} = U_{j,l_{k-1}} U_{l_{k-1},l_{k-2}} \cdots U_{l_2,l_1}$ *for* $k > 2$ *and* $\eta_{l_1} = 1$ *for* $l_1 = 1, \ldots, r$.

Projection techniques for bilinear systems can achieve the Volterra series interpolation if the matrices $V = \begin{bmatrix} v_1 & \cdots & v_r \end{bmatrix}$ and $W = \begin{bmatrix} w_1 & \cdots & w_r \end{bmatrix}$ are computed such that

$$v_j = \sum_{k=1}^{\infty} \sum_{l_1=1}^{r} \cdots \sum_{l_{k-1}=1}^{r} \eta_{l_1,\ldots,l_{k-1},j} (\sigma_j I - A)^{-1} N (\sigma_{l_{k-1}} I - A)^{-1} N \cdots N (\sigma_{l_1} I - A)^{-1} b,$$

$$w_j = \sum_{k=1}^{\infty} \sum_{l_1=1}^{r} \cdots \sum_{l_{k-1}=1}^{r} \eta_{l_1,\ldots,l_{k-1},j} (\sigma_j I - A)^{-T} N^T (\sigma_{l_{k-1}} I - A)^{-T} N^T \cdots N^T (\sigma_{l_1} I - A)^{-T} c^T,$$

for $j = 1, \ldots, r$. Clearly, it is not possible to compute explicitly the above infinite series of shifted systems for each v_j and w_j. However, it is shown in [8] that the above conditions are equivalent to solving the following generalized Sylvester equations for V and W

$$\begin{aligned} V\Lambda - AV - NVU^T + be^T &= 0, \\ WM - A^T W - N^T W S + c^T e &= 0, \end{aligned} \tag{17}$$

where $e \in \mathbb{R}^r$ is a vector of ones, Λ and M are $r \times r$ diagonal matrices with interpolation points as diagonal elements. U and S are $r \times r$ matrices with elements related to the weights of the Volterra series.

In case of $\mathcal{H}_2$-optimality, these conditions reduce to the interpolatory conditions of [5]. Based on the Volterra series interpolation, the authors in [8] also proposed the TB-IRKA algorithm that interpolates the truncated Volterra series where the sum over k is truncated.

3 Implicit Volterra series interpolation

In this section, we propose a new framework for the Volterra series interpolation by carefully selecting the weights of the series. We assume that the weights $\eta_{l_1,\ldots,l_{k-1},j}$ in (16) satisfy

$$\eta_{l_1,\ldots,l_{k-1},j} = \begin{cases} (\eta_{j,j})^{k-1}, & \text{if } j = l_{k-1} = \cdots = l_1, \\ 0, & \text{otherwise} \end{cases} \tag{18}$$

for $j = 1, \ldots, r$. If we also assume that $\|\eta_{j,j}(\sigma^{(j)} I - A)^{-1} N\| < 1$, then the following lemma shows that the Volterra series interpolation reduces to the interpolation of a linear system.

Lemma 3.1. *Let* $\eta_{l_1,\ldots,l_{k-1},j}$ *be the weights in the Volterra series interpolation satisfying* (18). *Furthermore,* $\|\eta_{j,j}(\sigma^{(j)} I - A)^{-1} N\| < 1$ *and* $\|\eta_{j,j}(\sigma^{(j)} I - \hat{A})^{-1} \hat{N}\| < 1$ *hold. Then, the Volterra series interpolation conditions in* (16) *become*

$$c(\sigma^{(j)} I - A - \eta_{j,j} N)^{-1} b =: G(\sigma^{(j)}) = \hat{G}(\sigma^{(j)}) := \hat{c}(\sigma^{(j)} I - \hat{A} - \eta_{j,j} \hat{N})^{-1} \hat{b}. \tag{19}$$

Proof. We need to show that under the given conditions,

$$\sum_{k=1}^{\infty}\sum_{l_1=1}^{r}\cdots\sum_{l_{k-1}=1}^{r}\eta_{l_1,\ldots,l_{k-1},j}H_k(\sigma^{(l_1)},\ldots,\sigma^{(l_{k-1})},\sigma^{(j)}) = G(\sigma^{(j)}). \tag{20}$$

We denote the left-hand side of the above equation by LHS and rewrite the series for a few values of k

$$LHS = \eta_j H_1(\sigma^{(j)}) + \sum_{l_1=1}^{r}\eta_{l_1,j}H_2(\sigma^{(l_1)},\sigma^{(j)}) + \sum_{l_1=1}^{r}\sum_{l_2=1}^{r}\eta_{l_1,l_2,j}H_3(\sigma^{(l_1)},\sigma^{(l_2)},\sigma^{(j)}) + \ldots.$$

Since the weights are only defined when $j = l_{k-1} = \cdots = l_1$, we use (18) to get

$$\begin{aligned}
LHS &= H_1(\sigma^{(j)}) + \eta_{j,j}H_2(\sigma^{(j)},\sigma^{(j)}) + (\eta_{j,j})^2 H_3(\sigma^{(j)},\sigma^{(j)},\sigma^{(j)}) + \ldots \\
&= c(\sigma^{(j)}I - A)^{-1}b + \eta_{j,j}c(\sigma^{(j)}I - A)^{-1}N(\sigma^{(j)}I - A)^{-1}b + \\
&\qquad (\eta_{j,j})^2 c\left((\sigma^{(j)}I - A)^{-1}N\right)^2(\sigma^{(j)}I - A)^{-1}b + \ldots \\
&= c\left[I + \eta_{j,j}(\sigma^{(j)}I - A)^{-1}N + \left(\eta_{j,j}(\sigma^{(j)}I - A)^{-1}N\right)^2 + \ldots\right](\sigma^{(j)}I - A)^{-1}b.
\end{aligned}$$

As $\|\eta_{j,j}(\sigma^{(j)}I - A)^{-1}N\| < 1$, we can use the Neumann lemma to obtain

$$LHS = c(I - \eta_{j,j}(\sigma^{(j)}I - A)^{-1}N)^{-1}(\sigma^{(j)}I - A)^{-1}b,$$

which reduces to the right-hand side of (20). Similarly

$$\sum_{k=1}^{\infty}\sum_{l_1=1}^{r}\cdots\sum_{l_{k-1}=1}^{r}\eta_{l_1,\ldots,l_{k-1},j}\hat{H}_k(\sigma^{(l_1)},\ldots,\sigma^{(l_{k-1})},\sigma^{(j)}) = \hat{G}(\sigma^{(j)}). \tag{21}$$

The relations (20) and (21) imply that (19) holds and this proves the lemma. □

Remark 3.1. *The assumption on the weights $\eta_{l_1,\ldots,l_{k-1},j}$ in (18) means that the matrix U in (16) is diagonal, i.e., $U = U_D := \operatorname{diag}(\eta_{1,1},\ldots,\eta_{r,r})$.*

Remark 3.2. *Although the assumption in (18) is simplifying our conditions of interpolation to a great deal, it is also imposing a condition on the original bilinear system in the sense that $\|\eta_{j,j}(\sigma^{(j)}I - A)^{-1}N\| < 1$ has to be ensured.*

We call this modified framework for interpolatory model reduction 'implicit Volterra series interpolation'.

Definition 3.1 (Implicit Volterra Series Interpolation)**.** *Let $S = \{\sigma^{(1)},\ldots,\sigma^{(r)}\}$ be a set of interpolation points and let the conditions in Lemma 3.1 hold. Then we say that a reduced bilinear system $\hat{\zeta}$ achieves the* implicit Volterra series interpolation *if the reduced state-space matrices fulfill*

$$c(\sigma^{(j)}I - A - \eta_{j,j}N)^{-1}b = \hat{c}(\sigma^{(j)}I - \hat{A} - \eta_{j,j}\hat{N})^{-1}\hat{b}. \tag{22}$$

Before going into the discussion of model order reduction via implicit Volterra series interpolation, we discuss the MIMO case.

4 The MIMO case

For MIMO systems, the Volterra series representation of the output $y(t)$ can be written as

$$y(t) = \sum_{k=1}^{\infty} \int_0^{t_1} \int_0^{t_2} \dots \int_0^{t_k} h_k(t_1, \dots, t_k) u(t - \sum_{i=1}^{k} t_i) \otimes \dots \otimes u(t - t_k) dt_k \dots dt_1. \quad (23)$$

The regular k-dimensional kernels are given as

$$\begin{aligned} h_1(t_1) &= Ce^{At_1}B, \\ h_2(t_1, t_2) &= Ce^{At_2}\bar{N}(I_m \otimes e^{At_1})(I_m \otimes B), \\ &\vdots \\ h_k(t_1, \dots, t_k) &= Ce^{At_k}\bar{N}(I_m \otimes e^{At_{k-1}})(I_m \otimes \bar{N}) \cdots \\ &\quad (\underbrace{I_m \otimes I_m \cdots \otimes I_m}_{k-2 \text{ times}} \otimes e^{At_2})(\underbrace{I_m \otimes I_m \cdots \otimes I_m}_{k-2 \text{ times}} \otimes \bar{N}) \\ &\quad (\underbrace{I_m \otimes I_m \cdots \otimes I_m}_{k-1 \text{ times}} \otimes e^{At_1})(\underbrace{I_m \otimes I_m \cdots \otimes I_m}_{k-1 \text{ times}} \otimes B), \end{aligned} \quad (24)$$

where $\bar{N} = \begin{bmatrix} N_1, \dots, N_p \end{bmatrix}$. The structure of the kernels for $k = 1$, $k = 2$ and $k \geq 3$ is clear from the above equations. Assuming that $x(0) = x_0 = 0$, the multivariable Laplace transform of the k-dimensional kernels can be written as

$$\begin{aligned} H_1(s_1) &= C(s_1 I - A)^{-1}B, \\ H_2(s_1, s_2) &= C(s_2 I - A)^{-1}\bar{N}(I_m \otimes (s_1 I - A)^{-1})(I_m \otimes B), \\ &\vdots \\ H_k(s_1, \dots, s_k) &= C(s_k I - A)^{-1}\bar{N}(I_m \otimes (s_{k-1} I - A)^{-1})(I_m \otimes \bar{N}) \cdots \\ &\quad (\underbrace{I_m \otimes I_m \cdots \otimes I_m}_{k-2 \text{ times}} \otimes (s_2 I - A)^{-1})(\underbrace{I_m \otimes I_m \cdots \otimes I_m}_{k-2 \text{ times}} \otimes \bar{N}) \\ &\quad (\underbrace{I_m \otimes I_m \cdots \otimes I_m}_{k-1 \text{ times}} \otimes (s_1 I - A)^{-1})(\underbrace{I_m \otimes I_m \cdots \otimes I_m}_{k-1 \text{ times}} \otimes B). \end{aligned} \quad (25)$$

The above expressions are commonly used for the input-output representation of MIMO bilinear systems, see [13] for details. However, with this representation, a model reduction technique can not ensure the Volterra series interpolation. This is because in the above representation, the size of the multivariable transfer matrix, $H_k(s_1, \dots, s_k)$, is $p \times m^k$, which changes with k.

We suggest to interpolate a closely related matrix function of fixed size $(p \times m)$ which is obtained by the sum of $p \times m$ blocks in a k-th order multivariable transfer matrix, $H_k(s_1, \dots, s_k)$. Each transfer matrix, $H_k(s_1, \dots, s_k)$ includes block diagonal matrices written in the form of Kronecker products which can be used to identify the $p \times m$ block matrices. Exemplarily we set $m = 2$. Then, the second subsystem becomes

$$\begin{aligned} H_2(s_1, s_2) &= C(s_2 I - A)^{-1} \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} (s_1 I - A)^{-1}B & 0 \\ 0 & (s_1 I - A)^{-1}B \end{bmatrix} \\ &= \begin{bmatrix} H_2^{(1)}(s_1, s_2) & H_2^{(2)}(s_1, s_2) \end{bmatrix}, \end{aligned} \quad (26)$$

where $H_2^{(1)}(s_1,s_2) = C(s_2I - A)^{-1}N_1(s_1I - A)^{-1}B$ and $H_2^{(2)}(s_1,s_2) = C(s_2I - A)^{-1}N_2(s_1I - A)^{-1}B$. This means that the output $y_2(s_1,s_2)$ for the two-dimensional subsystem is

$$\begin{aligned} y_2(s_1,s_2) &= H_2^{(1)}(s_1,s_2)u_1(s_1)u(s_2) + H_2^{(2)}(s_1,s_2)u_2(s_1)u(s_2) \\ &= \big(H_2^{(1)}(s_1,s_2)u_1(s_1) + H_2^{(2)}(s_1,s_2)u_2(s_1)\big)u(s_2). \end{aligned} \tag{27}$$

We assume that the output of the second subsystem is well approximated by the second subsystem of the reduced system if it interpolates the $p \times m$ matrix function

$$\mathcal{H}_2(s_1,s_2) = H_2^{(1)}(s_1,s_2) + H_2^{(2)}(s_1,s_2). \tag{28}$$

Notice that for $m > 2$, the above summation will include m matrices. Similarly, for the third subsystem, $\mathcal{H}_3(s_1,s_2,s_3)$ will include m^2 matrices and the k-th subsystem will correspond to m^{k-1} elements. The matrix function for the k-th subsystem therefore becomes

$$\mathcal{H}_k(s_1,\ldots,s_k) = \sum_{i_k=1}^{m^{k-1}} H_k^{(i_k)}(s_1,\ldots,s_k).$$

Now we can write the Volterra series interpolatory conditions for the matrix functions $\mathcal{H}_k(s_1,\ldots,s_k)$ for different values of k, similar to the SISO case. This means that the reduced system has to ensure

$$\begin{aligned} &\textstyle\sum_{k=1}^{\infty}\sum_{l_1=1}^{r}\cdots\sum_{l_{k-1}=1}^{r}\eta_{l_1,\ldots,l_{k-1},j}\,\mathcal{H}_k(\sigma^{(l_1)},\ldots,\sigma^{(l_{k-1})},\sigma^{(j)}) = \\ &\textstyle\sum_{k=1}^{\infty}\sum_{l_1=1}^{r}\cdots\sum_{l_{k-1}=1}^{r}\eta_{l_1,\ldots,l_{k-1},j}\,\hat{\mathcal{H}}_k(\sigma^{(l_1)},\ldots,\sigma^{(l_{k-1})},\sigma^{(j)}), \end{aligned} \tag{29}$$

for some weights, $\eta_{l_1,\ldots,l_{k-1},j}$ and interpolation points $\sigma^{(j)}$. The following theorem shows that the Volterra series interpolatory conditions can also be written as interpolatory conditions of a MIMO linear parametric system.

Theorem 4.1. *Let $\eta_{l_1,\ldots,l_{k-1},j}$ be the weights in the Volterra series interpolation satisfying (18) and the inequalities $\|\sum_{i=1}^{m}\eta_{j,j}(\sigma^{(j)}I - A)^{-1}N_i\| < 1$ and $\|\sum_{i=1}^{m}\eta_{j,j}(\sigma^{(j)}I - \hat{A})^{-1}\hat{N}_i\| < 1$ hold. Then the Volterra series interpolation conditions in (29) become*

$$C(\sigma^{(j)}I - A - \eta_{j,j}N)^{-1}B =: G(\sigma^{(j)}) = \hat{G}(\sigma^{(j)}) := \hat{C}(\sigma^{(j)}I - \hat{A} - \eta_{j,j}\hat{N})^{-1}\hat{B} \tag{30}$$

Proof. The result follows by using the weights in (18) in the interpolatory conditions (29) and using Neumann series, similar to the proof of Lemma 3.1. □

Both, Lemma 3.1 and Theorem 4.1, suggest that the interpolatory conditions of a bilinear system can be rewritten into interpolatory conditions of a linear parametric system, with parameters being the weights of the Volterra series. Next, we discuss two different techniques for linear parametric model reduction, that can be used to reduce the original bilinear system.

5 Linear Parametric Model Reduction

$\mathcal{H}_2$-optimal model reduction was addressed in the past years and approaches like B-IRKA or TB-IRKA were successfully applied to bilinear systems resulting from several applications. Two-sided projection via these algorithms requires the solution of two generalized Sylvester equations or four Lyapunov equations in every iteration step which makes the offline computational costs high for larger problem sizes. In this section, we consider approaches which are based on the observation that the Volterra series interpolation conditions lead to a linear system with an affine parameter dependence, see (30) for the MIMO case, (19) for SISO systems.

These linear, parametric systems can, in principle, be reduced by any method for parametric model order reduction. However, in this work, we further exploit another system's property which is encountered in many practical applications. Often, the bilinear terms are of low (numerical) rank ν, $N_i = b_i c_i^T$, with $b_i, c_i \in \mathbb{R}^{n\times\nu}$, such that we obtain a transfer function

$$G(s) = C(sI - A - \sum_{i=1}^{m} \eta_i \; b_i c_i^T)^{-1} B \tag{31}$$

corresponding to a linear, parametric system with low-rank variation in the state matrix,

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + \sum_{i=1}^{m} \eta_i \; b_i c_i^T x(t) + Bu(t), \\ y(t) & = & Cx(t). \end{array} \tag{32}$$

This system includes parameters $\eta_1, \ldots, \eta_m$ in a parameter domain $\mathcal{D}$. $\mathcal{D}$ has to be chosen such that the conditions in Lemma 3.1 (or in Theorem 4.1, respectively) are fulfilled. Methods for parametric model order reduction which are especially adapted to systems of this low-rank form will be described in the following two subsections.

5.1 Bilinear PMOR by system extension with $B(\eta)$

System (32) can be rewritten in an extended form by defining auxiliary inputs $\mathrm{u}_i(t) = c_i^T x(t)$, for $i = 1, \ldots, m$,

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \underbrace{[B,\, \eta_1 b_1,\, \ldots,\, \eta_m b_m\,]}_{B_{ext}(\eta)} \underbrace{\begin{pmatrix} u(t) \\ \mathrm{u}_1(t) \\ \vdots \\ \mathrm{u}_m(t) \end{pmatrix}}_{u_{ext}}, \\ \underbrace{\begin{pmatrix} y(t) \\ y_1(t) \\ \vdots \\ y_m(t) \end{pmatrix}}_{y_{ext}} &= \underbrace{\begin{bmatrix} C \\ c_1^T \\ \vdots \\ c_m^T \end{bmatrix}}_{C_{ext}} x(t) + \underbrace{\begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & -I & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \ldots & -I \end{bmatrix}}_{D_{ext}} \underbrace{\begin{pmatrix} u(t) \\ \mathrm{u}_1(t) \\ \vdots \\ \mathrm{u}_m(t) \end{pmatrix}}_{u_{ext}}. \end{aligned} \tag{33}$$

Then, if the artificial inputs $\{\mathrm{u}_1, \mathrm{u}_2, \ldots, \mathrm{u}_m\}$ are chosen in order to force the auxiliary outputs to zero, i.e., $y_1(t) = y_2(t) = \ldots = y_m(t) = 0$, the extended system (33) and the original parametric system (32) are equivalent.

This system can be reduced similarly to an approach based on IRKA which is described in [4]. The main observation here is that a combined $\mathcal{H}_2 \times \mathcal{L}_2$-norm of the system (33) is equivalent to the $\mathcal{H}_2$-norm of the non-parametric system $\mathcal{H}(s)L$ with

$$\mathcal{H}(s) \;=\; \begin{bmatrix} C \\ c_1^T \\ \vdots \\ c_m^T \end{bmatrix} (sI - A)^{-1}[B\, b_1 \ \ldots \ b_m] \tag{34}$$

and L being a Cholesky factor of

$$\int_{\mathcal{D}} \begin{bmatrix} I_m & 0 & \ldots & 0 \\ 0 & \eta_1 I_\nu & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \ldots & \eta_m I_\nu \end{bmatrix} \begin{bmatrix} I_m & 0 & \ldots & 0 \\ 0 & \eta_1 I_\nu & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \ldots & \eta_m I_\nu \end{bmatrix} d\eta \;= LL^T. \tag{35}$$

Thus, IRKA is used to reduce the dimension of $\mathcal{H}(s)L$ and the resulting reduced-order system is, on convergence, optimal w.r.t. a combined $\mathcal{H}_2 \times \mathcal{L}_2$-norm,

$$\begin{array}{rcl} \dot{\hat{x}}(t) &=& \hat{A}\,\hat{x}(t) \;+\; \hat{B}_{ext}(\eta)\,u_{ext}(t), \\ \hat{y}_{ext}(t) &=& \hat{C}_{ext}\,\hat{x}(t) \;+\; D_{ext}\,u_{ext}(t), \end{array} \tag{36}$$

with $\hat{A} = W^T A V$, $\hat{B}_{ext}(\eta) = W^T B_{ext}(\eta)$, $\hat{C}_{ext} = C_{ext} V$, and V, $W \in \mathbb{R}^{n \times r}$ for some reduced order $r \ll n$.

A reduced, parametric system of the same structure as (32) with $\hat{B} = W^T B$, $\hat{C} = C\,V$ and $\hat{b}_i = W^T b_i$, $\hat{c}_i^T = c_i^T V$, for $i = 1, \ldots, m$, can be recovered from (36) when $\mathrm{u}_1, \ldots, \mathrm{u}_m$ are chosen s.t. $\hat{y}_1(t) = \ldots = \hat{y}_m(t) = 0$:

$$\begin{array}{rcl} \dot{\hat{x}}(t) &=& \hat{A}\,\hat{x}(t) + \sum_{i=1}^m \eta_i\ \hat{b}_i\,\hat{c}_i^T\hat{x}(t) + \hat{B}u(t), \\ \hat{y}(t) &=& \hat{C}\hat{x}(t). \end{array} \tag{37}$$

A brief sketch of this approach is given in Algorithm 1.

Algorithm 1 extB(η): Bilinear PMOR by system extension with $B(\eta)$

1. Compute Cholesky factor L by calculating the integral (35) using appropriate parameter domain $\mathcal{D}$ (conditions in Lemma 3.1 for SISO systems or in Theorem 4.1 for MIMO systems are fulfilled).
2. Apply IRKA to $\mathcal{H}(s)L$ with $\mathcal{H}(s)$ as given in (34) to obtain V and W.
3. Reduced-order bilinear system (37) is obtained by projection:

$$\hat{A} = W^T A V, \quad \hat{B} = W^T B, \quad \hat{C} = CV,$$
$$\hat{b}_i = W^T b_i, \quad \hat{c}_i^T = c_i^T V, \quad \text{for } i = 1, \ldots, m.$$

5.2 Bilinear PMOR by system extension with $D(\eta)$

Alternatively, we can rewrite system (32) such that the parameter dependency is solely in D [3]:

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + \underbrace{[B,\, b_1,\, \ldots,\, b_m\,]}_{B_{ext}} \underbrace{\begin{pmatrix} u(t) \\ \mathrm{u}_1(t) \\ \vdots \\ \mathrm{u}_m(t) \end{pmatrix}}_{u_{ext}}, \\
\underbrace{\begin{pmatrix} y(t) \\ y_1(t) \\ \vdots \\ y_m(t) \end{pmatrix}}_{y_{ext}} &= \underbrace{\begin{bmatrix} C \\ c_1^T \\ \vdots \\ c_m^T \end{bmatrix}}_{C_{ext}} x(t) + \underbrace{\begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & -1/\eta_1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \ldots & -1/\eta_m \end{bmatrix}}_{D_{ext}(\eta)} \underbrace{\begin{pmatrix} u(t) \\ \mathrm{u}_1(t) \\ \vdots \\ \mathrm{u}_m(t) \end{pmatrix}}_{u_{ext}}.
\end{aligned} \tag{38}$$

Here, auxiliary inputs $\mathrm{u}_1(t) = \eta_1 c_1^T x(t), \ldots, \mathrm{u}_m(t) = \eta_m c_m^T x(t)$ are taken, and again, they should be chosen such that the auxiliary outputs are zero: $y_1(t) = y_2(t) = \ldots = y_m(t) = 0$.

Significantly, the parameterization of (38) resides entirely only in the feed-forward term $D_{ext}(\eta)$, which is matched exactly in many model order reduction approaches. Thus, we can apply IRKA [10] (or other non-parametric methods for MOR) to (38) to obtain an auxiliary reduced-order system similarly like in (36). The reduced system in original bilinear form can then be retrieved as in (37). The pseudo-code is listed in Algorithm 2.

Algorithm 2 extD(η): Bilinear PMOR by system extension with $D(\eta)$

1. Rewrite system (32) into extended form (38).
2. Apply IRKA to (38) to obtain V and W.
3. Reduced-order bilinear system (37) is obtained by projection:

$$\hat{A} = W^T A V, \quad \hat{B} = W^T B, \quad \hat{C} = CV,$$

$$\hat{b}_i = W^T b_i, \quad \hat{c}_i^T = c_i^T V, \quad \text{for } i = 1, \ldots, m.$$

6 Numerical Results

We use two different examples from the literature to demonstrate the efficiency of the new approach. These include the power system model and the heat transfer model. The systems are reduced by both techniques discussed in Section 5 and by B-IRKA and TB-IRKA for comparing the accuracy and the computational complexity of the reduction. Furthermore, we reduce both examples by applying linear IRKA to the linear part of the systems. The computed matrices V and W are then used to reduce the bilinear system as in (3).

The (B)-IRKA iterations are randomly initialized in all approaches.

6.1 Power System Model

The first example is a power system model [1] that is often used as a benchmark example for model reduction of bilinear systems. This system naturally has a low rank of one in all matrices $N_1, \ldots, N_4$ which constitute the bilinearity in the system. Thus, the methods extB(η) and extD(η) for parametric model order reduction can be applied. The size of the original model is 17 with 4 inputs and 3 outputs. Although the dimension is very low, and does not call for model order reduction in practice, we can use use it for verifying the behavior of the algorithms. We reduce the dimension to $r = 4, \ldots, 16$ with the extB(η) and extD(η) approaches and with BIRKA, TB-IRKA, and IRKA for comparison. The relative errors in the $\mathcal{H}_2$-norm are shown in Figure 1. We additionally compare in Figure 2 the offline times of the approaches considered.

It can be seen that the accuracy of the new approaches is nearly as good as in B-IRKA but with a lower computational complexity. TB-IRKA performs similarly well but also with higher offline costs. The naive approach of ignoring the bilinearity in the system's reduction is not surprisingly less accurate, but also cheap to compute. The peaks in Figure 2 indicate reduced orders for which the iteration of the respective method did not converge within the prescribed 100 maximum iterations steps.

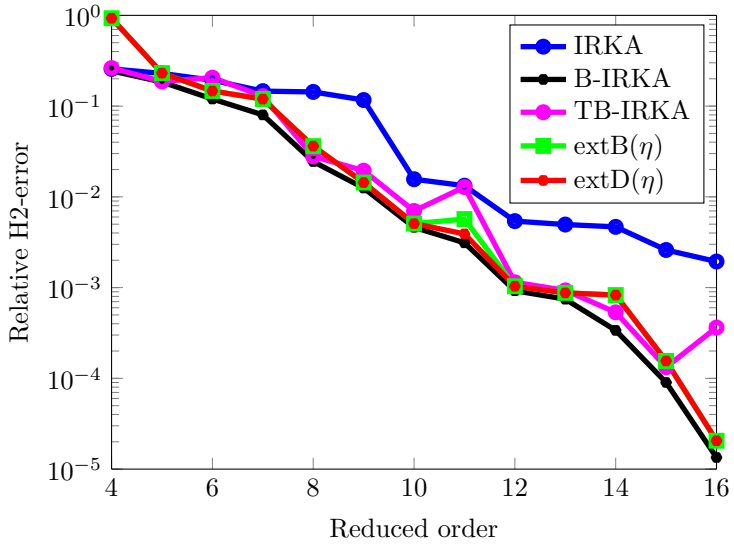

Figure 1: $\mathcal{H}_2$-errors for power system with different values of r.

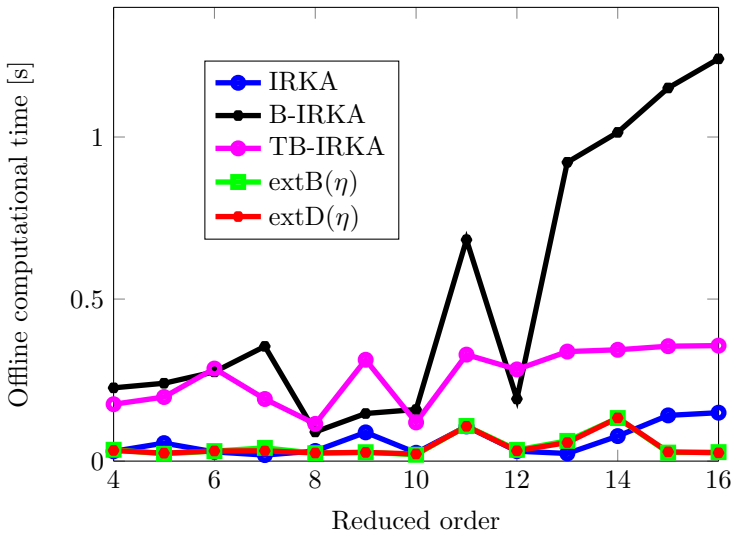

Figure 2: Offline times for power system with different values of r.

6.2 Heat Transfer Model

We consider a boundary controlled heat transfer system, that is a standard example for testing bilinear control systems [5, 6]. The dynamics of the system are governed by the heat equation

$$x_t = \Delta x, \qquad \text{on } \Omega = (0,1) \times (0,1), \tag{39}$$

with mixed Dirichlet and Robin boundary conditions

$$\begin{aligned} n \cdot \Delta x &= u(x-1), && \text{on } \Gamma_1, \\ x &= 0, && \text{on } \Gamma \setminus \Gamma_1, \end{aligned}$$

where Γ and Γ_1 are the boundaries of Ω and the heat transfer coefficient u is the input variable. In our example, the input acts only on a small part Γ_1 of the boundary such that the bilinear term is of rank one. The output of the system is chosen to be the average temperature on the grid. Spatial discretization with an equidistant grid and nodes x_{ij} (together with the boundary conditions) results in a SISO bilinear system of the form

$$\dot{x}(t) = Ax(t) + Nx(t)u(t) + Bu(t),$$

where $x = \mathrm{vec}(x_{ij})$. The input is set to $u(t) = \cos(\pi t)$. For details on the structure of the coefficient matrices, see for example [6]. The mesh size is set to $h = \frac{1}{36}$, so that the bilinear system has dimension $n = 2500$. We restrict the computations to a system of moderate dimension because of the costly computation of the $\mathcal{H}_2$-norm. The bilinear system is reduced to $r = 25$ by using implicit Volterra series interpolation with the extB(η) and extD(η) approaches. The results are compared with those computed by TB-IRKA and IRKA. The running time of B-IRKA was too long for this example. The relative $\mathcal{H}_2$-errors and the offline times can be found in Table 6.2. The transient responses of the original and the reduced systems and the corresponding relative output errors are shown in Figure 3 and 4, respectively.

It can be seen that extB(η) and extD(η) produce very accurate reduced-order systems in a very short time.

Table 1: Approximation errors and offline times for heat transfer system with $r = 25$.

	IRKA	TB-IRKA	extB(η)	extD(η)
Rel. $\mathcal{H}_2$-errors	1.6×10^{-3}	1.5×10^{-8}	2.1×10^{-8}	1.1×10^{-8}
Offline times [sec]	4.1	838.7	2.3	2.4

7 Conclusions

We proposed a new framework for projection-based model order reduction of bilinear systems. This framework is especially useful for bilinear systems with a bilinear part in low-rank form. Such systems can be rewritten into extended, parametric systems where the system's dimension can be reduced very efficiently by just applying IRKA.

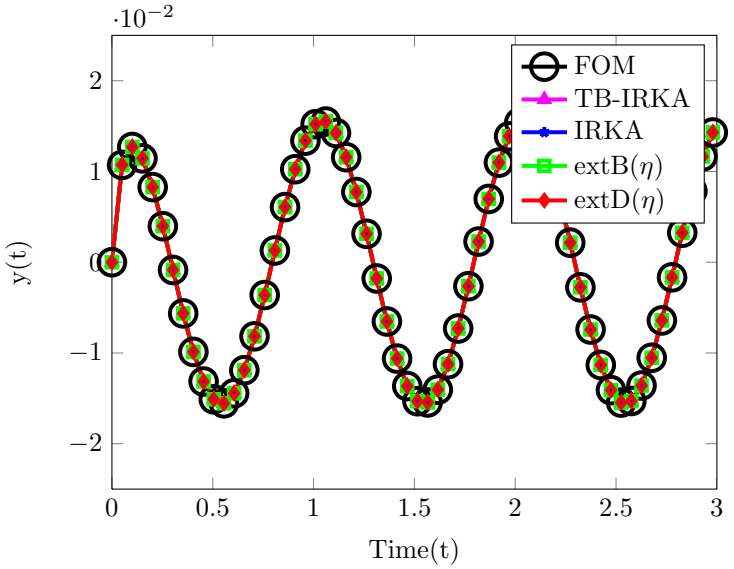

Figure 3: Transient response of heat transfer system with $r = 25$.

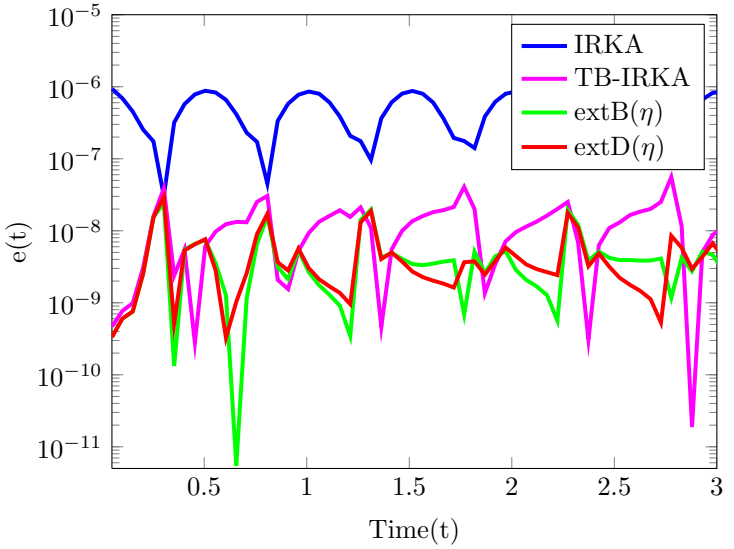

Figure 4: Relative output errors for heat transfer system with $r = 25$.

We have shown that the proposed approaches compute accurate reduced-order models involving linear computations only. This makes the framework well applicable to large-scale bilinear systems.

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