Parameterized Frequency-dependent Balanced Truncation for Model
Order Reduction of Linear Systems

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Abstract: Balanced truncation is the most commonly used model order reduction scheme in control engineering. This is
due to its favorable properties of automatic stability preservation and the existence of a computable error bound, enabling
the adaption of the reduced model order to a specified tolerance. It aims at minimizing the worst case error of the
frequency response over the full infinite frequency range. If a good approximation only over a finite frequency range is
required, frequency-weighted or frequency-limited balanced truncation variants can be employed. In this paper, we study
this finite-frequency model order reduction (FF-MOR) problem for linear time-invariant (LTI) continuous-time systems
within the framework of balanced truncation. Firstly, we construct a family of parameterized frequency-dependent
(PFD) mappings which transform the given LTI system to either a discrete-time or continuous-time PFD system. The
relationships between the maximum singular value of the given LTI system over pre-specified frequency ranges and the
maximum singular value of the PFD mapped systems over the entire frequency range are established. By exploiting
the properties of the discrete-time PFD mapped systems, a new parameterized frequency-dependent balanced truncation
(PFDBT) method providing a finite-frequency type error bound with respect to the maximum singular value of the error
systems is developed. Examples are included for illustration.

Key Words: Balanced truncation, Parameterized Frequency-dependent Balanced Truncation, KYP lemma, generalized
KYP lemma, Parameterized Frequency-dependent Bounded Real Lemma.

1 INTRODUCTION

Model order reduction (MOR) is an ubiquitous tool in the analysis and simulation of dynamical systems,
control design, circuit simulation, structural dynamics, computational fluid dynamics, and many more areas in the
computational sciences and engineering; see, e.g., [1–4].

Modeling of complex physical processes often leads to
dynamical systems with high-dimensional state-spaces, so
that the corresponding system is of large order \( n \). This may
lead to difficulties in the simulation, optimization, control
and design of such systems due to memory restrictions
and (run) time limitations for the execution of the related
algorithms. In general, the purpose of MOR is to produce
a lower dimensional system that has similar response
characteristics as the original system with far lower storage
requirements and largely reduced evaluation time. In
this paper, we focus on the MOR problem for linear
time-invariant (LTI) dynamical systems:

\[
G : \begin{cases}
\dot{x}(t) = Ax(t) + Bu(t) \\
y(t) = Cx(t) + Du(t)
\end{cases}
\]

where \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n}, D \in \mathbb{R}^{p \times m},
\)
\( x(t) \in \mathbb{R}^{n} \) is the state vector,
\( u(t) \in \mathbb{R}^{m} \) is the input
signal,
\( y(t) \in \mathbb{C}^{p} \) is the output signal. The imaginary unit
is denoted by \( j \), and \( \omega \in \mathbb{R} \) is related to the operating
frequency \( f \) (measured in Hertz) of the LTI system via
\( \omega = 2\pi f \). By abuse of notation, we denote the LTI system
as well as its transfer function by \( G(j\omega) \):

\[
G(j\omega) := C(j\omega I - A)^{-1}B + D
\]

A realization of the LTI system (1) is given by the matrix
tuple \((A, B, C, D)\). When appropriate, we will also use the
equivalent notation 
\[
\begin{bmatrix} A & B \\ C & D \end{bmatrix},
\]
which is common in control
theory.

The aim of MOR then is to approximate the LTI system (1)
by a reduced-order LTI system:

\[
G_r : \begin{cases}
\dot{x}_r(t) = A_r x_r(t) + B_r u(t) \\
y_r(t) = C_r x_r(t) + D_r u(t)
\end{cases}
\]

where \( A_r \in \mathbb{R}^{r \times r}, B_r \in \mathbb{R}^{r \times m}, C_r \in \mathbb{R}^{p \times r}, D_r \in \mathbb{R}^{p \times m},
\)
with \( r \ll n \), and so that \( y(t) \approx y_r(t) \) for \( t \) in some
chosen time range and for all admissible input functions
\( u(t) \). Similarly, we denote the transfer function of the
reduced system by \( G_r(j\omega) \):

\[
G_r(j\omega) := C_r(j\omega I - A_r)^{-1}B_r + D_r,
\]

In other words, in order to replace the original model
successfully, the reduced-order model should approximate
the input-output behaviors of the original system as well as possible. This underlying requirement on the reduced-order model means that the MOR problem inherently depends on the chosen class of input signals, that is, different types of input signals will lead to different MOR problems with respect to the approximation performance. From the frequency-domain viewpoint, signals can be classified into entire-frequency (EF) type signals and finite-frequency (FF) type signals, as listed in Table 1; cf. [5].

<table>
<thead>
<tr>
<th>Type</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>EF</td>
<td>$\omega \in \Omega = (-\infty, +\infty)$</td>
</tr>
<tr>
<td>LF</td>
<td>$\omega \in \Omega = [-\omega_l, +\omega_l]$</td>
</tr>
<tr>
<td>MF</td>
<td>$\omega \in \Omega_m = [\omega_1, +\omega_2]$</td>
</tr>
<tr>
<td>FF</td>
<td>$\omega \in \Omega_h = (-\infty, -\omega_h) \cup [\omega_h, +\infty)$</td>
</tr>
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</table>

Table 1: Different frequency ranges for input signals

Obviously, such a classification of the frequency range of input signals will give rise to several classes of MOR problems: EF-MOR when considering the full frequency range, and FF-MOR (including LF-MOR, MF-MOR, and HF-MOR) for limited frequency ranges, respectively. In case that there exists no a priori known frequency information of the input signals or the frequency of input signals belongs to a very wide range, EF-MOR problems will be the appropriate choice, and a uniform approximation performance over the entire frequency range should be taken into consideration [15]. For many practical cases, though, a certain range for the frequency of the input signals is pre-known. In these situations, it will be better to resort to a FF-MOR formulation since only the in-band input-output behavior of the original system is needed to be captured; cf., e.g., [6–8]. Thus, good in-band approximation performance can be expected while neglecting the out-of-band approximation performance, or, in other words, a better approximation quality in-band at the same reduced order is to be expected than for methods trying to approximate uniformly in the entire frequency band.

During the last decades, many efficient approaches such as balanced truncation [9, 10], moment matching [11, 12], and modal truncation [13] have been developed from different fields; see also the books [1–4] and the recent survey [14]. Among them, balanced truncation stands out for its beneficial properties relevant in control design, i.e., stability preservation and computable error bound, allowing for an automatic reduced-order model generation. Here, we focus on balanced truncation, and therefore in the following mainly review the literature with regard to attempts of adopting balanced truncation to the FF-MOR framework.

The idea underlying balanced truncation consists in transforming the state space system into a balanced form whose controllability and observability Gramians become diagonal and equal, together with a truncation of those states that are both difficult to reach and to observe. The standard version of balanced truncation is often called Lyapunov balancing (LyaBT), see, e.g., [15], and was first introduced by Moore in 1981 [9]. The reduced-order model obtained by LyaBT has diminishing error for increasing frequencies, but takes the maximum error often at $\omega = 0$. In order to match the DC gain, i.e., to have zero error at $\omega = 0$, but allowing a larger error at large frequencies, Liu and Anderson developed the singular perturbation approximation (SPA) scheme [16], which is also based on a balanced realization of the LTI system. Both, LyaBT and SPA, are widely appreciated and recognized as the most suitable techniques for EF-MOR problems since both of them provide a computable a priori error bound, called EF-type error bound in the following, with respect to the following entire-frequency approximation performance index:

$$\sigma_{\max}(G(j\omega) - G_r(j\omega)), \ \omega \in \Omega.$$  \hspace{1cm} (5)

Though this performance index related to the $H_\infty$-norm of the error system, is not minimized by LyaBT and SPA, the computed reduced-order models usually get close to optimal [1, 17]. The error bound makes it possible to choose the reduced order $r$ automatically. As mentioned above, LyaBT generally leads to good high-frequency approximation performance since the reduced-order models generated via LyaBT match the original model exactly at $|\omega| = \infty$, while SPA generally leads to good low-frequency approximation performance as the corresponding reduced-order models match the original model exactly at $\omega = 0$. However, it is unclear how good the in-band approximation performance over a specified HF (LF) range is, since only the EF-type error bound is known for LyaBT and SPA.

In order to make the standard LyaBT scheme more suitable for solving FF-MOR problems, several modified BT schemes have been developed. Frequency-weighted balanced truncation (FWBT) and frequency-limited Gramians balanced truncation (FGBT) are two popular ones for this purpose and were studied during the last 25 years. The common procedure of FWBT is to build a frequency-weighted model first by introducing input/output frequency weighted transfer functions and then apply the standard LyaBT or SPA procedure on the weighted model; see, e.g., [6, 18–23]. Indeed, good frequency-specific approximation performance may be obtained if the selected weighting function is appropriately chosen. However, the design iterations to search for such a weighting transfer function can be tedious and time consuming. Besides, FWBT also suffers from the drawback of the increased order of the weighted plant model. FGBT was first introduced by Gawronski and Juang in [24]. This methodology stems from the consideration of extending the definition of standard Gramians to the frequency-limited case and then applying the standard balanced truncation procedures to the frequency-limited Gramians [25–27]. An implementation of this method for truly large-scale systems was recently suggested in [28]. As has been pointed out in [15, 29],
FGBT may be infeasible in some cases as the solutions of the “frequency-limited Lyapunov equations” cannot be guaranteed to be positive semi-definite, and it provides no error bound. Although there exist several modified FGBT schemes, see, e.g., [15, 26] to overcome those drawbacks, good in-band performance generally cannot be guaranteed. More importantly, both FWBT and FGBT continue to use the EF-type index (5) to evaluate the actually desired finite-frequency approximation performance. This incompatibility between the intrinsic requirement and the achievement of the method yields many deficiencies. Since only EF-type error bounds are available, whether or not the in-band approximation performance has been improved cannot be guaranteed. In particular, FWBT and FGBT may give rise to poor in-band approximation performance together with a large error bound in some cases.

In [30], we studied the FF-MOR problem from the perspective of achieving good approximation quality locally by devising a balanced truncation style method satisfying an error bound at a prescribed frequency. The method shows good approximation quality locally in a neighborhood of the given frequency point, and this neighborhood is usually larger than for interpolatory (or moment-matching) methods that have zero error at the prescribed frequency. Nevertheless, this new method does not solve the FF-MOR problem satisfactorily as it provides no error bound valid on a (half-)finite interval.

The shortcomings of the approaches to adapt balanced truncation to the FF-MOR setting motivated us to study this problem from a new FF-type error bound centered viewpoint.

In this paper, we are dedicated to solving the FF-MOR problems within the framework of balanced truncation. In contrast to existing BT schemes, we are interested in developing a new way to provide in-band error bounds by using the following FF-type approximation performance index:

$$\sigma_{\text{max}}(G(j\omega) - G_r(j\omega)), \quad \omega \in \Omega_l/\Omega_m/\Omega_h. \quad (6)$$

Compared with the EF-type index (5), adopting the FF-type index (6) is obviously more appealing for FF-MOR problems. To this end, a fundamental tool estimating the maximum singular value of an LTI system over finite-frequency ranges is developed first, and then new BT based schemes are proposed for LF-MOR problems and HF-MOR problems. In particular, the contributions of this paper are:

1. By introducing an auxiliary user-defined parameter $\rho$, two kinds of discrete-time parameterized frequency-dependent (PFD) systems and two kinds of continuous-time parameterized frequency-dependent (PFD) systems are constructed by a suitable mapping applied to the given continuous-time LTI system. The mapping is determined with respect to the specified finite-frequency range. Furthermore, PFD bounded real lemmas bounding the maximum singular value of the given system over the pre-specified finite-frequency ranges are derived. It is shown that there exist special relationships between the maximum singular value of the given system over the pre-specified finite-frequency ranges and the maximum singular value of the PFD mapped systems over the entire frequency ranges.

2. By exploiting the standard discrete-time Lyapunov method and the developed PFD bounded real lemma, new PFD balanced truncation (PFDBT) schemes are proposed to solve the LF-MOR and HF-MOR problems, respectively. The new PFDBT methods generate reduced-order models and provide FF-type approximation error bounds in the sense of bounding the maximum singular value of the error system over the pre-specified frequency range.

It should be noticed that the detailed proofs and the results related to the HF case will be omitted in this conference paper due to space limitation, the full version including all the details can be accessed online [31] (arXiv:1602.04408).

**Notation:** For a matrix $M$, $M^T$ and $M^*$ denote its transpose and conjugate transpose, respectively. $M > 0$ and $M \geq 0$ indicate a positive definite and semi-definite matrix, respectively. The symbol $\star$ within a matrix represents symmetric entries and $He(M) := \frac{1}{2}(M + M^*)$ is the Hermitian part of a matrix $M$. $\sigma_{\text{max}}(G)$ denotes the maximum singular value of the transfer matrix $G$. $\Re(x)$ and $\Im(x)$ are the real and imaginary parts, respectively, of the complex scalar $x$.

### 2 Preliminaries

In this section, we will first review the well-known Kalman-Yakubovich-Popov (KYP) Lemma. The KYP Lemma [32] is a cornerstone for analyzing and synthesizing linear systems. In [5], Iwasaki and Hara successfully generalized the KYP Lemma from the entire-frequency case to different finite-frequency cases. The Generalized KYP Lemma and the KYP lemma will play a fundamental role in our development. Therefore, we state the original versions for continuous- and discrete-time LI systems in the following.

**Lemma 2.1 (Continuous-time KYP Lemma [32])**

Consider the linear continuous-time LTI system (1), and assume $(A, B)$ to be controllable as well as $A$ to have no eigenvalues on the imaginary axis. Given a matrix $\Pi \in \mathbb{R}^{n+m\times n+m}$, then the following statements are equivalent:

1. The frequency domain inequality

$$\begin{bmatrix} G^*(j\omega) & I \end{bmatrix} \Pi \begin{bmatrix} G(j\omega) & I \end{bmatrix} \leq 0 \quad \text{holds for all} \quad \omega \in (-\infty, +\infty). \quad (7)$$

2. There exists a symmetric matrix $P \geq 0$ such that the following linear matrix inequality holds:

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix} \begin{bmatrix} P & \ast \\ \ast & 0 \end{bmatrix} + \begin{bmatrix} C & D \\ I & 0 \end{bmatrix} \Pi \begin{bmatrix} C^T & D^T \\ 0 & I \end{bmatrix} \leq 0. \quad (8)$$
There exist a symmetric matrix $P > 0$ and matrices $K, L$ such that the following Lur'e matrix equation holds:

$$
\begin{bmatrix}
AB \\
I 0
\end{bmatrix}
\begin{bmatrix}
0 P \\
P 0
\end{bmatrix}
\begin{bmatrix}
AB \\
I 0
\end{bmatrix}^* + \begin{bmatrix}
CD \\
0 I
\end{bmatrix} \Pi \begin{bmatrix}
CD \\
0 I
\end{bmatrix}^* = -\begin{bmatrix}
LL^* & KL^* \\
KL^* & KK^*
\end{bmatrix}.
$$

(9)

Lemma 2.2 (Discrete-time KYP Lemma [32]) Consider a linear discrete-time system, realized by $(A, B, C, D)$, with transfer function $G(e^{j\theta})$, $(A, B)$ controllable, $A$ having no eigenvalues of modulus 1, and a matrix $\Pi \in \mathbb{R}^{n \times n}$. Then the following statements are equivalent:

a) The frequency domain inequality

$$
\begin{bmatrix}
G^*(e^{j\theta}) \\
I
\end{bmatrix} \Pi \begin{bmatrix}
G^*(e^{j\theta}) \\
I
\end{bmatrix} \leq 0 \quad \text{holds for all} \quad \theta \in \Theta : [-\pi, +\pi],
$$

(10)

b) There exists a symmetric matrix $P > 0$ such that the following linear matrix inequality holds:

$$
\begin{bmatrix}
AB \\
I 0
\end{bmatrix}
\begin{bmatrix}
0 P \\
P 0
\end{bmatrix}
\begin{bmatrix}
AB \\
I 0
\end{bmatrix}^* + \begin{bmatrix}
CD \\
0 I
\end{bmatrix} \Pi \begin{bmatrix}
CD \\
0 I
\end{bmatrix}^* \leq 0.
$$

(11)

c) There exist a symmetric matrices $P$ and $Q$ of appropriate dimensions, satisfying $Q > 0$ and

$$
\begin{bmatrix}
AB \\
I 0
\end{bmatrix}
\begin{bmatrix}
-Q \\
P \omega^2 Q
\end{bmatrix}
\begin{bmatrix}
AB \\
I 0
\end{bmatrix}^* + \begin{bmatrix}
CD \\
0 I
\end{bmatrix} \Pi \begin{bmatrix}
CD \\
0 I
\end{bmatrix}^* \leq 0.
$$

(12)

The generalized versions of the KYP Lemma for Low frequency case introduced by Iwasaki and Hara read as follows:

Lemma 2.3 (Continuous-time generalized KYP lemma [5]) Under the assumptions of Lemma 2.1, the following statements are equivalent:

a) The frequency domain inequality

$$
\begin{bmatrix}
G^*(j\omega) \\
I
\end{bmatrix} \Pi \begin{bmatrix}
G^*(j\omega) \\
I
\end{bmatrix} \leq 0 \quad \text{holds for all} \quad \omega \in \Omega_l/\Omega_{st}/\Omega_h.
$$

(13)

b) There exist symmetric matrices $P$ and $Q$ of appropriate dimensions, satisfying $Q > 0$ and

$$
\begin{bmatrix}
AB \\
I 0
\end{bmatrix}
\begin{bmatrix}
-Q \\
P \omega^2 Q
\end{bmatrix}
\begin{bmatrix}
AB \\
I 0
\end{bmatrix}^* + \begin{bmatrix}
CD \\
0 I
\end{bmatrix} \Pi \begin{bmatrix}
CD \\
0 I
\end{bmatrix}^* \leq 0.
$$

(14)

Remark 2.4 The main role of the KYP and GKYP lemmas is to characterize various system properties in terms of an inequality condition on the Popov function corresponding to the LTI system over the entire frequency range or over finite frequency ranges. In case the matrix $\Pi$ in (7), (10) is specialized as in the common bounded-realness case: $\Pi_{BR} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, the (generalized) KYP lemma is referred to as (generalized) bounded real lemma or (generalized) positive real lemma. Actually, the EF-type index (5) could be equivalently characterized by the entire-frequency inequality (7) by choosing $\Pi = \Pi_{BR}$. Similarly, the FF-type index (6) could be equivalently characterized by the finite-frequency inequality (10) by choosing $\Pi = \Pi_{BR}$.

For more details about the KYP and GKYP lemmas, we refer the reader to [5, 32].

3 Main Results

3.1 PFD Mapped Systems and PFD Bounded-Real Lemma (MF & LF Cases)

In this subsection, we first define a family of PFD mapped systems for a given system with respect to a pre-specified MF or LF range, then present the derived PFD bounded real lemma to show the relationships between the entire-frequency maximum singular value of the PFD mapped systems and the MF maximum singular value of the given system. Noticing that the LF range can be viewed as a special case of the MF range by letting $\overline{\omega}_1 = -\overline{\omega}_1$ and $\overline{\omega}_2 = \overline{\omega}_2$, where the different frequencies in the LF and MF cases are defined in Table 1, all the definitions and results will be presented in the more general MF setting.

Definition 3.1 (PFD Mapped Systems (LF & MF Cases)) Let $(A, B, C, D)$ be a realization of the LTI system (1), $\rho \in \mathbb{R}$, and $\overline{\omega}_c = (\overline{\omega}_1 + \overline{\omega}_2)/2$, $\overline{\omega}_d = (\overline{\omega}_2 - \overline{\omega}_1)/2$ with $\overline{\omega}_1, \overline{\omega}_2$ defining the considered finite frequency range as in Table 1. Then we define the following PFD mapped systems corresponding to (1):

a) The discrete-time system $\hat{G}_{m\rho c}(e^{j\theta}) := \begin{bmatrix}
\hat{A}_{m\rho c} & \hat{B}_{m\rho c} \\
\hat{C}_{m\rho c} & \hat{D}_{m\rho c}
\end{bmatrix}$ is constructed via the following upper type PFD mapping:

$$
\begin{bmatrix}
\hat{A}_{m\rho c} \\
\hat{B}_{m\rho c}
\end{bmatrix} = \begin{bmatrix}
A + \rho I \\
B + \rho C
\end{bmatrix}, \begin{bmatrix}
\hat{C}_{m\rho c} \\
\hat{D}_{m\rho c}
\end{bmatrix} = \begin{bmatrix}
C + \rho \overline{\omega}_c I \\
D + \rho \overline{\omega}_d I
\end{bmatrix},
$$

and

$$
\hat{G}_{m\rho c} \text{ will be referred to as upper type PFD mapped system w.r.t. the MF range } \Omega_{m\rho}.
$$

b) The discrete-time system $\hat{G}_{m\rho c}(e^{j\theta}) := \begin{bmatrix}
\hat{A}_{m\rho c} & \hat{B}_{m\rho c} \\
\hat{C}_{m\rho c} & \hat{D}_{m\rho c}
\end{bmatrix}$ is constructed via the following lower type PFD mapping:

$$
\begin{bmatrix}
\hat{A}_{m\rho c} \\
\hat{B}_{m\rho c}
\end{bmatrix} = \begin{bmatrix}
A - \rho I \\
B - \rho C
\end{bmatrix}, \begin{bmatrix}
\hat{C}_{m\rho c} \\
\hat{D}_{m\rho c}
\end{bmatrix} = \begin{bmatrix}
C - \rho \overline{\omega}_c I \\
D - \rho \overline{\omega}_d I
\end{bmatrix},
$$

and

$$
\hat{G}_{m\rho c} \text{ will be referred to as lower type PFD mapped system w.r.t. the MF range } \Omega_{m\rho}.
$$
c) The continuous-time system $G_{mpc}(\omega) := \begin{bmatrix} A_{mpc} & B_{mpc} \\ C_{mpc} & D_{mpc} \end{bmatrix}$ is constructed via the following left type PFD mapping:

$$(A_{mpc}, B_{mpc}, C_{mpc}, D_{mpc}) = \mathscr{M}_{mpc}(A, B, C, D, \Omega_m),$$

where

$$A_{mpc} = -\frac{1}{2} I - (\rho - j\varpi_d)(j\varpi I - A)^{-1},$$

$$B_{mpc} = (j\varpi I - A)^{-1} B,$$

$$C_{mpc} = C(j\varpi I - A)^{-1},$$

$$D_{mpc} = -(\rho - j\varpi_d)(C(j\varpi I - A)^{-1} B + D),$$

$$G_{mpc}$$ will be referred to as left type PFD mapped system w.r.t. the MF range $\Omega_m$.

d) The continuous-time system $G_{mpc2}(\omega) := \begin{bmatrix} A_{mpc2} & B_{mpc2} \\ C_{mpc2} & D_{mpc2} \end{bmatrix}$ is constructed via the following right type PFD mapping:

$$(A_{mpc2}, B_{mpc2}, C_{mpc2}, D_{mpc2}) = \mathscr{M}_{mpc2}(A, B, C, D, \Omega_m),$$

where

$$A_{mpc2} = -\frac{1}{2} I - (\rho + j\varpi_d)(j\varpi I - A)^{-1},$$

$$B_{mpc2} = (j\varpi I - A)^{-1} B,$$

$$C_{mpc2} = C(j\varpi I - A)^{-1},$$

$$D_{mpc2} = (\rho + j\varpi_d)(C(j\varpi I - A)^{-1} B + D),$$

$$G_{mpc2}$$ will be referred to as right type PFD mapped system w.r.t. the MF range $\Omega_m$.

**Proposition 3.2** Letting

$$\rho^* = \max\left(\frac{\varpi^2_d}{\tilde{\omega}} - 9\text{e}(\lambda_i)^2 - (\varpi_c + 3\text{m}(\lambda_i))^2}{2\pi\text{e}(\lambda_i)}\right), i = 1, 2, ..., n,$$

(19)

where $\lambda_i$, $i = 1, 2, ..., n$ are the eigenvalues of the matrix $A$, then the following statements are true:

a) If $\rho > \rho^*_m$, then the matrix $\hat{A}_{mpc}$ is Schur stable.

b) If $\rho < -\rho^*_m$, then the matrix $\hat{A}_{mpc}$ is Schur stable.

c) If $\rho > \rho^*_m$, then the matrix $\hat{A}_{mpc}$ is Hurwitz stable.

d) If $\rho > \rho^*_m$, then the matrix $\hat{A}_{mpc2}$ is Hurwitz stable.

**Proof.** With the upper case PFD mapping (15), the eigenvalues $\hat{\lambda}_{mpc}$, $i = 1, ..., n$ of the mapped matrix $\hat{A}_{mpc}$ are:

$$\hat{\lambda}_{mpc} = (\rho^2 + \varpi^2_d)^{\frac{1}{2}}(\rho + j\varpi_c - \lambda_i)^{-1}.$$  

If $\rho > \rho^*_m$, we have $\hat{\lambda}_{mpc} < 1$, $i = 1, ..., n$. Thus the matrix $\hat{A}_{mpc}$ is Schur stable, which implies a).

Similarly, the statements b)–d) can be proved by inspecting the eigenvalues of the mapped matrices. ■

**Theorem 3.3** (PFD Bounded-Real Lemma (LF& MF Case)) Denote the entire-frequency range ($\theta \in [-\pi, +\pi]$) in the discrete-time setting as $\Theta$, and use $\Omega$ and $\Omega_m$ to represent the entire-frequency range and middle-frequency range (see Table I), respectively. The following statements on the relationship between the maximum singular value of the mapped systems over the entire-frequency range and the maximum singular value of the given system are true:

a) If $\sigma_{\max}(G_{mpc}(\theta)) \leq \gamma_{mpc}$, $\forall \theta \in \Theta$, then $\sigma_{\max}(G(\theta)) \leq (\rho^2 + \varpi^2_d)^{\frac{1}{2}} \gamma_{mpc}$, $\forall \varpi \in \Omega_m$.

b) If $\sigma_{\max}(G_{mpc}(\theta)) \leq \gamma_{mpc}$, $\forall \theta \in \Theta$, then $\sigma_{\max}(G(\theta)) \leq (\rho^2 + \varpi^2_d)^{\frac{1}{2}} \gamma_{mpc}$, $\forall \varpi \in \Omega_m$.

c) If $\sigma_{\max}(G_{mpc}(\theta)) \leq \gamma_{mpc}$, $\forall \varpi \in \Omega$, then $\sigma_{\max}(G(\theta)) \leq (\rho^2 + \varpi^2_d)^{\frac{1}{2}} \gamma_{mpc}$, $\forall \varpi \in \Omega_m$.

d) If $\sigma_{\max}(G_{mpc2}(\theta)) \leq \gamma_{mpc2}$, $\forall \varpi \in \Omega$, then $\sigma_{\max}(G(\theta)) \leq (\rho^2 + \varpi^2_d)^{\frac{1}{2}} \gamma_{mpc2}$, $\forall \varpi \in \Omega_m$.

**Proof.** To simplify the notation, we will henceforth denote $(\rho I + j\varpi_c I - A)$ as $\alpha_{pc}$ as:

a) Since $\sigma_{\max}(\hat{G}_{mpc}(\theta)) \leq \gamma_{mpc}$ $\forall \theta \in \Theta$ is equivalent to

$$\left[\begin{array}{ccc} I & 0 \\ 0 & -\gamma_{mpc} I \end{array}\right] \left[\begin{array}{ccc} \hat{G}_{mpc}(\theta) \\ I \end{array}\right] \leq 0 \quad \forall \theta \in \Theta.$$  

(20)

According to the discrete-time KYP lemma (Lemma 2.2), there exist a positive symmetric matrix $P_{mpc}$ and two matrices $L_{mpc}, K_{mpc}$ satisfying

$$\begin{align*}
\hat{A}_{mpc} P_{mpc} A_{mpc}^* - P_{mpc} + B_{mpc} D_{mpc} &= -L_{mpc} L_{mpc}^*, \\
A_{mpc} P_{mpc} C_{mpc}^* + B_{mpc} D_{mpc}^* &= -L_{mpc} K_{mpc}^*, \\
\hat{C}_{mpc} P_{mpc} C_{mpc}^* + D_{mpc} D_{mpc}^* - \gamma_{mpc} I &= -K_{mpc} K_{mpc}^*. 
\end{align*}$$  

(21a)  

(21b)  

(21c)

Now define $Q = \hat{P}_{mpc}$, $P = \rho \hat{P}_{mpc}$. From the above equations (21a)–(21c), we obtain the following identities: First,

$$-\text{He}(\varpi_c I - A)Q(j\varpi_c I - A)) + AP + PA^* + BB^*$$

$$= -(\rho I + j\varpi_c I - A)P_{mpc}(\rho I + j\varpi_c I - A)^{-1} + \varpi^2_d P_{mpc}$$

$$- \rho(j\varpi_c I - A)P_{mpc} - \rho P_{mpc}(\rho I + j\varpi_c I - A)^{-1} + BB^*$$

$$= (\rho^2 + \varpi^2_d)\hat{P}_{mpc} - \alpha_{pc} P_{mpc} \alpha_{pc}^* + BB^*$$

$$= \alpha_{pc}^* \hat{P}_{mpc} \alpha_{pc} - \alpha_{pc} P_{mpc} \alpha_{pc}^* + BB^*.$$  

(22)

where $L = (\rho I + j\varpi_c I - A)L_{mpc}$. Next, we have

$$\begin{align*}
(j\varpi_c I - A)Q C^* + PC^* + BB^* &= (\varpi_c I - A)Q C^* + \rho P_{mpc} C^* + BB^* \\
&= (\rho^2 + \varpi^2_d)\hat{P}_{mpc} - \alpha_{pc}^* \hat{P}_{mpc} \alpha_{pc} - \alpha_{pc} P_{mpc} \alpha_{pc}^* + BB^* \\
&= \alpha_{pc} L_{mpc} K_{mpc} - L_{mpc}^* K_{mpc} + L_{mpc}^* C + BB^* \\
&= \alpha_{pc} L_{mpc} K_{mpc} - L_{mpc}^* K_{mpc} + L_{mpc}^* C. 
\end{align*}$$  

(23)  

$$= -\alpha_{pc} L_{mpc} \left(\rho^2 + \varpi^2_d\right)^{\frac{1}{2}} K_{mpc} - C L_{mpc}^*.$$  

(24)
where $K = (\rho^2 + \omega_0^2)^{\frac{1}{2}} K_{mpc} - C L_{mpc}$. Last, we obtain

\[-CQC^* + DD^* - (\rho^2 + \omega_0^2)^{\frac{1}{2}} \gamma_{mpc} I\]

\[(21a) = -C \left( (\rho^2 + \omega_0^2)^{\frac{1}{2}} P_{mpc} \rho_{mpc}^{*} + \omega_{mpc}^2 L_{mpc}^{*} \right) C^* + DD^* - (\rho^2 + \omega_0^2)^{\frac{1}{2}} \gamma_{mpc} I\]

\[(21b) = (\rho^2 + \omega_0^2) \left( C_{mpc} \rho_{mpc}^{*} C_{mpc}^{*} + D_{mpc} \rho_{mpc}^{*} - \gamma_{mpc}^2 I \right) + (\rho^2 + \omega_0^2)^{\frac{1}{2}} C \left( A_{mpc} P_{mpc} C_{mpc}^{*} + B_{mpc} D_{mpc}^{*} \right) + (\rho^2 + \omega_0^2)^{\frac{1}{2}} \left( A_{mpc} P_{mpc} C_{mpc}^{*} + B_{mpc} D_{mpc}^{*} \right) C^* + \omega_{mpc}^2 L_{mpc}^{*} C^* + \omega_{mpc}^2 L_{mpc}^{*} C^* + \omega_{mpc}^2 L_{mpc}^{*} C^* = -KK^*.

(25)

Combing (22)–(25) yields

\[\begin{bmatrix} A & I \\ C & 0 \end{bmatrix} \begin{bmatrix} -Q + \omega_{mpc} Q + P & 0 \\ -\omega_{mpc} Q & -C^* \end{bmatrix} \begin{bmatrix} A & I \\ C & 0 \end{bmatrix}^* + \begin{bmatrix} B & 0 \\ D & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -\omega_{mpc}^2 I \end{bmatrix} \begin{bmatrix} B & 0 \\ D & I \end{bmatrix}^* = \begin{bmatrix} -LL^* + LL^* \\ -KK^* \end{bmatrix}.

According to the GKY lemma (Lemma 2.3), the following inequality can be concluded:

\[G^*(j\omega) \begin{bmatrix} I & 0 \\ 0 & -\omega_{mpc}^2 I \end{bmatrix} \begin{bmatrix} G^*(j\omega) \\ I \end{bmatrix} \leq 0 \quad \forall \omega \in \Omega_m.

This leads to

\[\sigma_{\text{max}}(G(j\omega)) \leq (\rho^2 + \omega_0^2)^{\frac{1}{2}} \gamma_{mpc} \quad \forall \omega \in \Omega_m,

showing statement a).

b) Since $\sigma_{\text{max}}(G_{mpc}(j\omega)) \leq \gamma_{mpc} \quad \forall \theta \in \Theta$ is equivalent to

\[G_{mpc}(j\omega) \begin{bmatrix} I & 0 \\ 0 & -\omega_{mpc}^2 I \end{bmatrix} \begin{bmatrix} G_{mpc}(j\omega) \\ I \end{bmatrix} \leq 0 \quad \forall \theta \in [-\pi, +\pi].

(26)

According to the discrete-time KYP lemma (Lemma 2.2), there exist a positive symmetric matrix $P_{mpc}$ and two matrices $L_{mpc}, K_{mpc}$ satisfying

\[A_{mpc} P_{mpc} A_{mpc}^{*} - P_{mpc} + B_{mpc} B_{mpc}^{*} = -L_{mpc}^{*} L_{mpc} \\
(27a)

A_{mpc} P_{mpc} C_{mpc}^{*} + B_{mpc} D_{mpc}^{*} = -L_{mpc}^{*} L_{mpc} \\
(27b)

C_{mpc} P_{mpc} C_{mpc}^{*} + D_{mpc} D_{mpc}^{*} - \gamma_{mpc}^2 I = -K_{mpc}^{*} K_{mpc} \\
(27c)

Now define $Q = P_{mpc}$ and $P = \rho P_{mpc}$. From the above equations (27a)–(27c), we obtain the following identities: First,

\[-\text{He}(j\omega I - A)Q(j\omega I - A) + AP + PA^{*} + BB^{*}

\[-\rho(j\omega I - A)P_{mpc} + \rho P_{mpc}(j\omega I - A)^{\frac{1}{2}} + \omega_{mpc}^2 P_{mpc}^{2} + \rho P_{mpc}(j\omega I - A)^{\frac{1}{2}} + BB^{*}

\[(16) = (\rho^2 + \omega_0^2) \omega_{mpc}^2 (j\omega I - A)^{\frac{1}{2}} \left( A_{mpc} P_{mpc} A_{mpc}^{*} - P_{mpc} + B_{mpc} B_{mpc}^{*} \right) (j\omega I - A)^{\frac{1}{2}}

\[(28) = -LL^*.

According to the GKY lemma (Lemma 2.3), the following inequality holds:

\[G^*(j\omega) \begin{bmatrix} I & 0 \\ 0 & -\omega_{mpc}^2 I \end{bmatrix} \begin{bmatrix} G^*(j\omega) \\ I \end{bmatrix} \leq 0 \quad \forall \omega \in \Omega_m.

(32)
This leads to
\[
\sigma_{\text{max}}(G(j\omega)) \leq \left(\rho^2 + \frac{\gamma_m^2}{\delta}\right)\gamma_{mpc} \quad \forall \omega \in \Omega_m, \quad (33)
\]
showing statement b).

c) \(\sigma_{\text{max}}(G_{I1}(j\omega)) \leq \gamma_{I1} \quad \forall \omega \in \Omega\) is equivalent to
\[
\begin{bmatrix} G^*(j\omega) \\ I \end{bmatrix} \Pi \begin{bmatrix} G^*(j\omega) \\ I \end{bmatrix} \leq 0 \quad \forall \omega \in \Omega. \quad (34)
\]

According to the GKYP lemma (Lemma 2.3), there exist a positive symmetric matrix \(P_{mp1}\) and two matrices \(L_{mp1}, K_{mp1}\) satisfying
\[
\begin{align*}
A_{mp1}P_{mp1} + P_{mp1}A_{mp1}^* + B_{mp1}B_{mp1}^* &= -L_{mp1}L_{mp1}^*, \quad (35a) \\
P_{mp1}C_{mp1}^* + B_{mp1}D_{mp1} &= -L_{mp1}K_{mp1}, \quad (35b) \\
D_{mp1}K_{mp1} - \gamma_{mp1}^2I &= -K_{mp1}K_{mp1}^*, \quad (35c)
\end{align*}
\]
Now define \(Q = P_{mp1}\) and \(P = \rho P_{mp1}\). From the above equations (35a)–(35c), we obtain the following identities: First,
\[
\begin{align*}
& \quad -\text{He}((j\omega I - A)Q(j\omega I - A)) + AP + PA^* + BB^*
\quad = -((j\omega I - A)P_{mp1}(j\omega I - A) + \rho P_{mp1})^*
\quad - \rho((j\omega I - A)P_{mp1} - P_{mp1}(j\omega I - A))^* + BB^*
\quad \overset{(36)}{=} (j\omega I - A) \left( A_{mp1}P_{mp1} - P_{mp1}A_{mp1}^* + B_{mp1}B_{mp1}^* \right) (j\omega I - A)^*
\quad \overset{(35a)}{=} -LL^*
\end{align*}
\]
where \(L = (j\omega I - A)L_{mp1}\). Next, we have
\[
\begin{align*}
(j\omega I - A)(Q(j\omega I - A)) + PC^* + BD^* &= (j\omega I - A)P_{mp1}C_{mp1}^* + \rho P_{mp1}C_{mp1}^* + BD^*
\quad \overset{(35a)}{=} (j\omega I - A)(j\omega I - A)^{-1}BB^*(j\omega I - A)^{-1}
\quad - (\rho + \gamma_{mp1}^2)(j\omega I - A)^{-1}P_{mp1}
\quad - (\rho + \gamma_{mp1}^2)(j\omega I - A)^{-1}P_{mp1}
\quad - (j\omega I - A)(j\omega I - A)^{-1}L_{mp1}L_{mp1}^C
\quad + BD^* + (\rho + \gamma_{mp1}^2)P_{mp1}C_{mp1}^*
\quad \overset{(36)}{=} (j\omega I - A)L_{mp1} \left( -(\rho + \gamma_{mp1}^2)K_{mp1} - CL_{mp1} \right)^*
\quad = -LK^*
\end{align*}
\]
where \(K = -(\rho - \gamma_{mp1}^2)K_{mp1} - CL_{mp1}\). Last, we obtain
\[
\begin{align*}
-CQ^* + DD^* - (\rho^2 + \frac{\gamma_m^2}{\delta})\gamma_{mp1}^2I
\quad = -CP_{mp1}C_{mp1}^* + DD^* - (\rho^2 + \frac{\gamma_m^2}{\delta})\gamma_{mp1}^2I
\quad \overset{(35a)}{=} -C(j\omega I - A)^{-1}BB^*(j\omega I - A)^{-1}
\quad - (\rho - \gamma_{mp1}^2)(j\omega I - A)^{-1}P_{mp1}
\quad - P_{mp1}(j\omega I - A)^{-1}(\rho - \gamma_{mp1}^2)^* + L_{mp1}L_{mp1}^C
\quad + DD^* - \gamma^2I
\quad \overset{(35a)}{=} -C(j\omega I - A)^{-1}BB^*(j\omega I - A)^{-1}C^*
\quad + C(j\omega I - A)^{-1}BB^*(j\omega I - A)^{-1}C^*
\quad + DD^*(j\omega I - A)^{-1}C^* - (\rho - \gamma_{mp1}^2)K_{mp1}L_{mp1}^C
\quad + (j\omega I - A)^{-1}BB^*(j\omega I - A)^{-1}C^*
\quad + (j\omega I - A)^{-1}BB^*(j\omega I - A)^{-1}C^*
\quad - CL_{mp1}L_{mp1}^C
\quad = -CL_{mp1}L_{mp1}^C
\quad \overset{(36)}{=} (j\omega I - A)(P_{mp1} - P_{mp1}^* + B_{mp1}B_{mp1}^*)
\quad = -(\rho + \gamma_{mp1}^2)(j\omega I - A)^{-1}L_{mp1}L_{mp1}^C
\quad + BD^* + (\rho + \gamma_{mp1}^2)P_{mp1}C_{mp1}^*
\quad \overset{(36)}{=} (j\omega I - A)L_{mp1} \left( -(\rho + \gamma_{mp1}^2)K_{mp1} - CL_{mp1} \right)^*
\quad = -LK^*
\end{align*}
\]
where \(L = (j\omega I - A)L_{mp1}\). Next, we have
\[
\begin{align*}
& \quad -\text{He}((j\omega I - A)Q(j\omega I - A)) + AP + PA^* + BB^*
\quad = -((j\omega I - A)P_{mp1}(j\omega I - A) + \rho P_{mp1})^*
\quad - \rho((j\omega I - A)P_{mp1} - P_{mp1}(j\omega I - A))^* + BB^*
\quad \overset{(36)}{=} (j\omega I - A) \left( A_{mp1}P_{mp1} - P_{mp1}A_{mp1}^* + B_{mp1}B_{mp1}^* \right) (j\omega I - A)^*
\quad \overset{(35a)}{=} -LL^*
\end{align*}
\]
Now define \(Q = P_{mp1}\) and \(P = \rho P_{mp1}\). From the above equations (35a)–(35c), we obtain the following identities: First,
\[
\begin{align*}
& \quad -\text{He}((j\omega I - A)Q(j\omega I - A)) + AP + PA^* + BB^*
\quad = -((j\omega I - A)P_{mp1}(j\omega I - A) + \rho P_{mp1})^*
\quad - \rho((j\omega I - A)P_{mp1} - P_{mp1}(j\omega I - A))^* + BB^*
\quad \overset{(36)}{=} (j\omega I - A) \left( A_{mp1}P_{mp1} - P_{mp1}A_{mp1}^* + B_{mp1}B_{mp1}^* \right) (j\omega I - A)^*
\quad \overset{(35a)}{=} -LL^*
\end{align*}
\]
where
\[ L = (j\varpi_2 I - A)I_{m}, \]
\[ K = (-\rho + j\varpi_2)I_{m}. \]

According to the GKYP lemma (Lemma 2.3), the following inequality holds:
\[
\left[ G^*(j\omega) \right]^* \left[ \begin{array}{c}
-1 & 0 \\
0 & \sigma^2 + \varpi_2^2
\end{array} \right] G^*(j\omega) \leq 0 \quad \forall \omega \in \Omega_m.
\]

This leads to
\[
\sigma_{\max}(G(j\omega)) \leq (\rho^2 + \varpi_2^2)^{1/2} \gamma_{m}, \quad \forall \omega \in \Omega_m.
\]

Remark 3.4 The linear matrix inequality defined in the Generalized Kalman-Yakovich-Popov lemma [5] (in particular, the generalized bounded real lemma) is a necessary and sufficient criteria on checking the finite-frequency maximum singular value. In contrast, the PFD bounded real lemma only provides a conservative estimation of the maximum singular value over the specified frequency range. However, the PFD bounded real lemma makes it feasible to analyze the finite-frequency maximum singular value via the standard Kalman-Yakovich-Popov Lemma (in particular, the standard bounded real lemma [32]), in which a simpler linear matrix inequality requiring less matrix decision variables is involved. Moreover, the PFD bounded real lemma paves a way to solve some finite-frequency problems (such as FF-MOR) by exploiting the existing entire-frequency techniques.

Remark 3.5 It should be noted that the parameter matrices of all kinds of PFD mapped systems generally will be complex matrices for the general MF cases (i.e. \( \varpi_c \neq 0 \)). For the LF cases (i.e. \( \varpi_c = 0 \)), the parameter matrices of the upper and lower type discrete-time PFD mapped systems are real if the parameter matrices of the given system \( G(j\omega) \) are real.

3.2 PFD Balanced Truncation (LF Case)

Based upon the above preliminaries and results, we now are at the stage to present the PFDBT algorithm for the LF case.

Theorem 3.6 (LF-type error bound via LF case PFDBT)

Given a linear continuous-time system \( G(j\omega) \) and a pre-known LF interval \( \Omega = [-\varpi, +\varpi] \). Suppose the reduced model \( \hat{G}(j\omega) \) is generated via the LF case PFDBT algorithm, then the approximation performance over the pre-specified frequency interval satisfies the following FF-type error bound:

\[
\sigma_{\max}(G(j\omega) - \hat{G}(j\omega)) \leq 2(\rho^2 + \varpi^2)^{1/2} \sum_{i=r+1}^{n} \sigma_i, \quad \forall \omega \in \Omega
\]

Algorithm 1 PFDBT (LF Case)

Input: Full-order model \( (A, B, C, D) \), frequency interval \( \Omega = [-\varpi, +\varpi] \), user-defined admissible parameter \( \rho \) and the order of the reduced model \( r \).

Alternative 1.

apply the standard discrete-time Lyapunov-BT (see, e.g., [23, 33, 34]) to the mapped discrete-time system \( \hat{G}_{m,pc}(e^\theta) \) to obtain the discrete-time reduced model \( \hat{G}_{m,pc}(e^\theta) = (\hat{A}_{m,pc}, \hat{B}_{m,pc}, \hat{C}_{m,pc}, \hat{D}_{m,pc}) \). Compute the reduced-order model by applying the inverse upper type PFD mapping as follows:

\[
\hat{A}_r = (pI + j\varpi_c I) - (\rho^2 + \varpi_c^2)^{1/2} \hat{A}_{m,pc}^{-1},
\hat{B}_r = (pI + j\varpi_c I - \hat{A}_r)\hat{B}_{m,pc},
\hat{C}_r = \hat{C}_{m,pc}(pI + j\varpi_c I - \hat{A}_r),
\hat{D}_r = \hat{D}_{m,pc} - \hat{C}_r(pI + j\varpi_c I - \hat{A}_r)^{-1}\hat{B}_r.
\]

where \( \varpi_c = 0 \) and \( \varpi_d = \varpi \).

Alternative 2.

apply the standard discrete-time Lyapunov-BT to the discrete-time PFD mapped system \( \hat{G}_{m,pc}(e^\theta) \), obtain the discrete-time mapped reduced model \( \hat{G}_{m,pc}(e^\theta) = (\hat{A}_{m,pc}, \hat{B}_{m,pc}, \hat{C}_{m,pc}, \hat{D}_{m,pc}) \) Compute the reduced-order model by applying the inverse upper type PFD mapping as follows:

\[
\hat{A}_r = -j\varpi_c I - \varpi_d (\rho^2 + 1)^{1/2}(-\hat{A}_{m,pc})^{-1},
\hat{B}_r = (\rho^2 + 1)^{1/2}(j\varpi_c I - \hat{A}_r)\hat{B}_{m,pc},
\hat{C}_r = (\rho^2 + 1)^{1/2}\hat{C}_{m,pc}(j\varpi_c I - \hat{A}_r),
\hat{D}_r = (\rho^2 + 1)^{1/2}\varpi_d\hat{D}_{m,pc} - \hat{C}_r(j\varpi_c I - \hat{A}_r)^{-1}\hat{B}_r.
\]

where \( \varpi_c = 0 \) and \( \varpi_d = \varpi \).

Output: Reduced-order model:

\[ G_r(j\omega) = (\hat{A}_r, \hat{B}_r, \hat{C}_r, \hat{D}_r). \]

Proof. The error system between the original high-order system \( G(j\omega) \) and the truncated \((n - 1)\)th reduced system \( G_r(j\omega) \) can be represented by

\[ E_r(j\omega) = G(j\omega) - G_r(j\omega) = \begin{bmatrix} A_r & 0 & \hat{B}_r \\ \hat{C}_r & \hat{D}_r \end{bmatrix} = \begin{bmatrix} A & 0 & B \\ -C_r & D - D_r \end{bmatrix}. \]

Suppose the parameter matrices \( (A_r, B_r, C_r, D_r) \) are computed via the Alternative 1 of Algorithm 2 and apply the upper case PFD mapping to the error system (50), the mapped error system can be represented by

\[
\hat{E}_{m,pc}(\theta) = \hat{G}_{m,pc}(e^\theta) - \hat{G}_{m,pc}(e^\theta) = \begin{bmatrix} \hat{A}_{m,pc} & 0 & \hat{B}_{m,pc} \\ 0 & \hat{A}_{m,pc} & \hat{B}_{m,pc} \end{bmatrix} = \begin{bmatrix} \hat{A}_{m,pc} & 0 & \hat{B}_{m,pc} \\ -\hat{C}_{m,pc} & \hat{D}_{m,pc} - \hat{D}_{m,pc} \end{bmatrix}.
\]

Since \( \hat{G}_{m,pc}(e^\theta) \) is the reduced model obtained by applying the standard discrete-time Lyapunov-BT to the upper PFD mapped system \( \hat{G}_{m,pc}(e^\theta) \). According to the results of well-known standard Lyapunov-BT [9, 23, 33, 34], we
have
\[ \sigma_{\max}(G_{\text{mpc}}(e^{\rho j\omega}) - G_{\text{mpc}}(e^{\rho j\omega}))) \leq 2 \sum_{i=r+1}^{n} \sigma_{i}, \quad \forall \rho \in \Theta. \]  
(52)

Noticing that the error system \( E_{\text{mpc}}(e^{\rho j\omega}) \) can be obtained by applying the upper type PFD mapping on the error system \( E_{r}(e^{\rho j\omega}) \), then we have
\[ \sigma_{\max}(G(e^{\rho j\omega}) - G(e^{\rho j\omega})) \leq 2(\rho^2 + \omega^2)^{\frac{1}{2}} \sum_{i=r+1}^{n} \sigma_{i}, \quad \forall \omega \in \Omega_{1}. \]
according to Theorem 3.3. Thus, the proof is completed.

Remark 3.7 To the best of our knowledge, this is the first result that provides an FF-type error bound in the framework of balanced truncation. Similar with the EF-type error bound provided by LyaBT (see, e.g., [23], [33], [34]), the FF-type error bound (49) is also very simple and a priori. Comparing the values of the EF-type error bound with the FF-type error bound theoretically is difficult, however, it is shown that the FF-type error bound could be smaller than the EF-type error bound by choosing a proper parameter \( \rho \). To obtain a proper value of the parameter \( \rho \), we suggest a simple line search over the admissible range of \( \rho \). As shown by the examples in the sequel, one can find the proper parameter by observing the curves of the FF-type error bound with respect to several different values of the parameter \( \rho \). How to compute the optimal parameter rendering the FF-type error bound as the smallest value is still an open problem for further investigation.

Remark 3.8 It is well-known that the original model is required to be stable to apply the standard LyaBT, moreover, the stability will be preserved by the reduced model generated via the standard LyaBT. The stability restriction on the original models is not needed for PFDBT. For non-stable original model, one could apply the PFDBT just by choosing a large enough parameter rendering the PFD mapped matrices \( \mathbf{A}_{\text{mpc}}(\rho) \) or \( \mathbf{A}_{\text{mpc}}(\rho) \) Schur stable. At the same time, the PFDBT does not possess the stability preservation property. In other words, the stability of the reduced model cannot be theoretically guaranteed even when the original model is stable. According to our numerical experiments, one could always obtain a stable reduced model whenever the original model is stable by selecting a proper parameter (especially by letting the parameter \( \rho \) be large enough).

Remark 3.9 In algorithm 2, only the discrete-time PFD mapped systems and the discrete-time LyaBT procedures are involved. Obviously, if we resort to the continuous-time PFD mapped systems and the continuous-time LyaBT procedures in a similar way, another routines producing reduced models could be derived. Unfortunately, the parameter matrices of the reduced models generally will become complex matrices under in this case. Besides, extending the PFDBT for the LF case to the MF case is also feasible. Likewise, such an extension generally will leads to complex reduced models since \( \omega_c \neq 0 \).

4 Illustrating Examples

In this section we demonstrate the validity of the PFD bounded real lemma and the advantages of the PFDBT schemes through three examples.

Example 4.1 Consider a simple linear continuous-time system (1) with the following parameter matrices:
\[
\begin{bmatrix}
A \\
B \\
C \\
D
\end{bmatrix} =
\begin{bmatrix}
-4.1859 & 0.7195 & 1.8712 \\
1.7797 & -1.1872 & 1.1639 \\
0.4528 & -2.4099 & 2.5606
\end{bmatrix}.
\]  
(54)

We are interested to apply the proposed PFD bounded real lemma for estimating the maximum singular value of this system over different low-frequency ranges:
\[ \Omega_{1}^{\text{L}} := [-0.1, 0.1], \quad \Omega_{1}^{\text{F}} := [-1, 1], \quad \Omega_{2}^{\text{L}} := [-10, 10], \quad \Omega_{2}^{\text{F}} := [-100, 100]. \]

Figure 1: Example 1, estimating the maximum singular value of given system over specified frequency range via PFD bounded real lemma

As shown in Figure 1, the estimated maximum singular values obtained by the PFD bounded real lemma with any admissible parameter \( \rho \) are always larger than the actual maximum singular values over the specified low-frequency ranges. In particular, the gaps between the estimated maximum singular values and the actual maximum singular value may be very small if the adjustable parameter \( \rho \) lies in an appropriate range. The results indicate the validity and effectiveness of the proposed PFD bounded real lemma.

Example 4.2 Let us consider LF-MOR of a 6th order linear continuous-time system (the parameter matrices can be found in the full arXiv version [31]) over two different frequency ranges \( \Omega_{1}^{\text{L}} := [-1, 1] \) and \( \Omega_{2}^{\text{L}} := [-2, 2] \). In order to show the differences between the standard LyaBT, SPA and the proposed PFDBT, the EF-type error bound via LyaBT(1), the FF-type error bound via PFDBT as well as the actual approximation error are depicted in Figure 2 and Figure 3, respectively.

To apply the proposed PFDBT, here we just randomly choose three different admissible values of the parameter \( \rho \) (\( \rho_1 = 4, \rho_2 = 7, \rho = 20 \)). As Figures 2 and 3 illustrate, the
The proposed PFDBT performs better than the standard LyaBT. In particular, the actual in-band approximation error resulting from PFDBT also can be smaller than the actual in-band error obtained by SPA, which is well-known for its good low-frequency approximation performance. More importantly, the PFDBT possesses an advantage on the in-band approximation error. Obviously, the EF-type error bounds provided by PFDBT are smaller than the EF-type error bound provided by LyaBT(SPA). This property makes the proposed PFDBT more appealing for selecting the minimum order of the reduced model satisfying an a priori given error tolerance.

Example 4.3 (The CD player benchmark example [35])

This original model of the benchmark CD player example describes the dynamics between a swing arm on which a lens is mounted by means of two horizontal leaf springs. The model has 120 states, i.e., $n = 120$ (Please refer to [35] for more details). Suppose the interesting frequency ranges are of low-frequency type. Here we aim to compare the achievable in-band error bound by applying the standard LyaBT and the proposed PFDBT.

Given four different low-frequency ranges $\Omega^i \delta := \{1, 2, 3, 4\}$, the corresponding EF-type error bounds with different values $(\rho^i, 10\rho^i, 100\rho^i, i = 1, 2, 3, 4)$ of the adjustable parameter are depicted in Figure 4, where $\rho^i$, $i = 1, 2, 3, 4$, is the minimum value rendering the PFD mapped system $\hat{G}_{\text{map}}(\omega)$ Schur stable. For comparison, the EF-type error bounds obtained by standard LyaBT are also included. From Figure 4, it is clear that the PFDBT method is able to produce a smaller in-band error. Certainly, to what extend the in-band error bound can be improved depends on the choice of the parameter $\rho$.

5 Conclusions and Future Work

5.1 Conclusions

In this paper, we have constructed a family of PFD mapped systems and established the connections between the maximum singular values over different frequency ranges (i.e. entire-frequency range and finite-frequency range) of the PFD mapped systems and the original system in the form of PFD bounded real lemma. Furthermore, new parameterized frequency-dependent balanced truncation (PFDBT) schemes solving some finite frequency MOR problems were developed, by which a finite-frequency type error bound could be provided. Numerical examples were carried out to illustrate the theoretical results.

5.2 Future Work

While this paper has demonstrated the potential of the PFD bounded real lemma as well as its application to the FF-MOR problem (i.e., PFDBT schemes), many opportunities for extending the scope of this paper exist. Here we presents some of these directions.

1) Bounded realness is known as one special case of the general quadratic performance in linear system analysis. It would be interesting to further construct more general PFD mappings with respect to general quadratic performance conditions by following the same methodology utilized in this paper. Accordingly, it is desired to develop PFD type KYP Lemma for dealing with the general finite-frequency quadratic performance analysis as well as bridging the well-known KYP lemma and the generalized KYP lemma.

2) For many other synthesis problems (such as controller design, filter design, etc), the KYP Lemma and the generalized KYP Lemma play a fundamental role in the existing solutions. It is expected that one can develop some
alternative solutions for many finite-frequency synthesis problems by exploiting the PFD bounded real lemma in this paper as well as the possible PFD type KYP lemma to be developed in the future.

3) In consideration of the practical implementation of the reduced model, the PFDBT schemes proposed in this paper (also see its arXiv version [31]) only solve LF-MOR and HF-MOR problems well. Further extensions and improvement of the PFDBT method should be developed to generate reduced models with real parameter matrices for solving the MF-MOR problems.

REFERENCES