MINIMAL REALIZATION AND MODEL REDUCTION OF STRUCTURED SYSTEMS

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Numerical Analysis and Scientific Computing Seminar
Courant Institute, NYU
September 13, 2019

Supported by:
1. Introduction

2. Minimal Realization

3. Reachability and Observability for SLS

4. Model Order Reduction

5. Numerical Results

6. Outlook and Conclusions
1. Introduction
   Model Reduction of Linear Systems
   Structured Linear Systems
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   Existing Approaches

2. Minimal Realization

3. Reachability and Observability for SLS

4. Model Order Reduction

5. Numerical Results

6. Outlook and Conclusions
Original System \((E = I_n)\)

\[ \Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t). \end{cases} \]

- states \(x(t) \in \mathbb{R}^n\),
- inputs \(u(t) \in \mathbb{R}^m\),
- outputs \(y(t) \in \mathbb{R}^p\).

Goals:

\[ \|y - \hat{y}\| < \text{tolerance} \cdot \|u\| \text{ for all admissible input signals.} \]
Model Reduction of Linear Systems
Linear Time-Invariant (LTI) Systems

Original System ($E = I_n$)

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- states $x(t) \in \mathbb{R}^n$,
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $y(t) \in \mathbb{R}^p$.

Reduced-Order Model (ROM)

$\hat{\Sigma} : \begin{cases} \dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t), \\ \hat{y}(t) = \hat{C}\hat{x}(t) + \hat{D}u(t). \end{cases}$

- states $\hat{x}(t) \in \mathbb{R}^r$, $r \ll n$
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $\hat{y}(t) \in \mathbb{R}^p$.

Goals:

$\|y - \hat{y}\| < \text{tolerance} \cdot \|u\|$ for all admissible input signals.
### Model Reduction of Linear Systems

**Linear Time-Invariant (LTI) Systems**

**Original System** \((E = I_n)\)

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- states \(\hat{x}(t) \in \mathbb{R}^r, r \ll n\)
- inputs \(u(t) \in \mathbb{R}^m\),
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Goals:

\[||y - \hat{y}|| < \text{tolerance} \cdot ||u||\] for all admissible input signals.

Secondary goal: reconstruct approximation of \(x\) from \(\hat{x}\).
Application of **Laplace transform** \( (x(t) \mapsto x(s), \dot{x}(t) \mapsto sx(s) - x(0)) \) to LTI system

\[
\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)
\]

with \( x(0) = 0 \) yields:

\[
sx(s) = Ax(s) + Bu(s), \quad y(s) = Cx(s) + Du(s),
\]
**Linear Systems in Frequency Domain**

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sx(s) = Ax(s) + Bu(s), \quad y(s) = Cx(s) + Du(s),
\]

\[\implies\] I/O-relation in frequency domain:

\[
y(s) = \left( C(sI_n - A)^{-1} B + D \right) u(s).
\]

\(H(s)\) is the **transfer function** of \(\Sigma\).
Linear Systems in Frequency Domain

Application of Laplace transform \( (x(t) \mapsto x(s), \dot{x}(t) \mapsto sx(s) - x(0)) \) to LTI system

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\( H(s) \) is the transfer function of \( \Sigma \).

Model reduction in frequency domain: Fast evaluation of mapping \( u \to y \).
Formulating model reduction in frequency domain

Approximate the dynamical system

\[
\begin{align*}
\dot{x} &= Ax + Bu, \quad A \in \mathbb{R}^{n \times n}, \ B \in \mathbb{R}^{n \times m}, \\
y &= Cx + Du, \quad C \in \mathbb{R}^{p \times n}, \ D \in \mathbb{R}^{p \times m},
\end{align*}
\]

by reduced-order system

\[
\begin{align*}
\dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}u, \quad \hat{A} \in \mathbb{R}^{r \times r}, \ \hat{B} \in \mathbb{R}^{r \times m}, \\
\hat{y} &= \hat{C}\hat{x} + \hat{D}u, \quad \hat{C} \in \mathbb{R}^{p \times r}, \ \hat{D} \in \mathbb{R}^{p \times m}
\end{align*}
\]

of order \( r \ll n \), such that

\[
\|y - \hat{y}\| = \|Hu - \hat{Hu}\| \leq \|H - \hat{H}\| \cdot \|u\| < \text{tolerance} \cdot \|u\|.
\]

\[\implies \text{Approximation problem:} \quad \min_{\text{order } (\hat{H}) \leq r} \|H - \hat{H}\|,
\]

where, mostly, \( \| \cdot \| = \| \cdot \|_{\mathcal{H}_\infty} \text{ or } \| \cdot \| = \| \cdot \|_{\mathcal{H}_2} \).
Second-order / mechanical / vibrational systems:

\[ M\ddot{x}(t) + L\dot{x}(t) + Kx(t) = Bu(t), \quad y(t) = C_p x(t) + C_v \dot{x}(t). \]
Second-order / mechanical / vibrational systems:

\[ M\ddot{x}(t) + L\dot{x}(t) + Kx(t) = Bu(t), \quad y(t) = C_p x(t) + C_v \dot{x}(t). \]

Apply Laplace transform \( \mapsto \)

\[ s^2 Mx(s) + sLx(s) + Kx(s) = Bu(s), \quad y(s) = C_p x(s) + sC_v x(s) \]
Second-order / mechanical / vibrational systems:

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Apply Laplace transform \( \Laplace \):

\[ s^2 Mx(s) + sLx(s) + Kx(s) = Bu(s), \quad y(s) = C_p x(s) + sC_v x(s) \]

\[ \Rightarrow y(s) = (C_p + sC_v)(s^2 M + sL + K)^{-1} Bu(s) =: C(s)K(s)^{-1} B(s) u(s) \]
Second-order / mechanical / vibrational systems:

\[ M \ddot{x}(t) + L \dot{x}(t) + Kx(t) = Bu(t), \quad y(t) = C_p x(t) + C_v \dot{x}(t). \]

Apply Laplace transform \( \Rightarrow \)

\[ s^2 M x(s) + s L x(s) + K x(s) = B u(s), \quad y(s) = C_p x(s) + s C_v x(s) \]

\[ \Rightarrow y(s) = (C_p + s C_v)(s^2 M + s L + K)^{-1} B u(s) =: C(s) K(s)^{-1} B(s) u(s) \]

Time-delay systems:

\[ E \dot{x}(t) = A_1 x(t) + A_2 x(t - \tau) + B u(t), \quad y(t) = C x(s) \]
Second-order / mechanical / vibrational systems:

\[ M\ddot{x}(t) + L\dot{x}(t) + Kx(t) = Bu(t), \quad y(t) = C_p x(t) + C_v \dot{x}(t). \]

Apply Laplace transform \( \mapsto \)

\[ s^2 Mx(s) + sLx(s) + Kx(s) = Bu(s), \quad y(s) = C_p x(s) + sC_v x(s) \]

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Time-delay systems:

\[ Ex(t) = A_1 x(t) + A_2 x(t - \tau) + Bu(t), \quad y(t) = Cx(s) \]

Apply Laplace transform \( \mapsto \)

\[ sEx(s) = A_1 x(s) + e^{-\tau s} A_2 x(s) + Bu(s), \quad y(s) = Cx(s) \]
Second-order / mechanical / vibrational systems:

\[ M \ddot{x}(t) + L \dot{x}(t) + Kx(t) = Bu(t), \quad y(t) = C_p x(t) + C_v \dot{x}(t). \]

Apply Laplace transform \( \rightarrow \)

\[ s^2 Mx(s) + sLx(s) + Kx(s) = Bu(s), \quad y(s) = C_p x(s) + sC_v x(s) \]

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\[ sEx(s) = A_1 x(s) + e^{-\tau s} A_2 x(s) + Bu(s), \quad y(s) = Cx(s) \]

\[ \implies y(s) = C(sE - A_1 - e^{-\tau s} A_2)^{-1}Bu(s) =: C(s)K(s)^{-1}B(s)u(s) \]
Second-order / mechanical / vibrational systems:

\[ M\ddot{x}(t) + L\dot{x}(t) + Kx(t) = Bu(t), \quad y(t) = C_p x(t) + C_v \dot{x}(t). \]

Apply Laplace transform \( \Rightarrow \)

\[ s^2 Mx(s) + sLx(s) + Kx(s) = Bu(s), \quad y(s) = C_p x(s) + sC_v x(s) \]

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Other examples: integro-differential / fractional systems, systems with surface loss, 1D PDE control, \ldots Note: all systems are linear w.r.t. the mapping \( u \rightarrow y \)!
Consider **Structured Linear System (SLS)** in frequency domain, using general set-up:

\[
H(s) = C(s)K(s)^{-1}B(s),
\]

where

\[
C(s) = \sum_{i=1}^{\ell_\gamma} \gamma_i(s)C_i, \quad K(s) = sE - \sum_{i=1}^{\ell_\alpha} \alpha_i(s)A_i, \quad B(s) = \sum_{i=1}^{\ell_\beta} \beta_i(s)B_i,
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- with \( E, A_i \in \mathbb{R}^{n \times n}, B_i \in \mathbb{R}^{n \times m} \), and \( C_i \in \mathbb{R}^{p \times n} \), and \( \alpha_i(s), \beta_i(s) \) and \( \gamma_i(s) \) are meromorphic functions.
Consider **Structured Linear System (SLS)** in frequency domain, using general set-up:

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- For simplicity, in this talk \( p = m = 1 \) (SISO case).
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- We assumed that \(E\) is invertible (no descriptor behavior).
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1) **First-order systems:** \( C(s) = C, \ B(s) = B, \) and \( K(s) = sE - A. \)
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1) **First-order systems:** \(C(s) = C, B(s) = B,\) and \(K(s) = sE - A.\)
2) **Second-order systems:** \(C(s) = C_p + sC_v, B(s) = B,\) and \(K(s) = s^2M + sL + K.\)
Consider **Structured Linear System (SLS)** in frequency domain, using general set-up:

\[ H(s) = C(s)K(s)^{-1}B(s), \]  

where

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3) **Time-delay systems:** \( C(s) = C, \) \( B(s) = B, \) and \( K(s) = sE - A_1 - A_2e^{-s\tau}. \)
Consider **Structured Linear System (SLS)** in frequency domain, using general set-up:

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$$

- with $E, A_i \in \mathbb{R}^{n \times n}, B_i \in \mathbb{R}^{n \times m},$ and $C_i \in \mathbb{R}^{p \times n},$ and $\alpha_i(s), \beta_i(s)$ and $\gamma_i(s)$ are meromorphic functions.
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3) **Time-delay systems:** $C(s) = C, B(s) = B,$ and $K(s) = sE - A_1 - A_2e^{-s\tau}.$
4) **EM w/ surface loss:** $C(s) = sB, B(s) = B,$ and $K(s) = s^2M + sL + K - \frac{1}{\sqrt{s}}N.$
Consider **Structured Linear System (SLS)** in frequency domain, using general set-up:

$$H(s) = C(s)K(s)^{-1}B(s),$$

where

$$C(s) = \sum_{i=1}^{\ell_{\gamma}} \gamma_i(s)C_i, \quad K(s) = sE - \sum_{i=1}^{\ell_{\alpha}} \alpha_i(s)A_i, \quad B(s) = \sum_{i=1}^{\ell_{\beta}} \beta_i(s)B_i,$$

- with $E, A_i \in \mathbb{R}^{n \times n}, B_i \in \mathbb{R}^{n \times m}$, and $C_i \in \mathbb{R}^{p \times n}$, and $\alpha_i(s), \beta_i(s)$ and $\gamma_i(s)$ are meromorphic functions.
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3) **Time-delay systems:** $C(s) = C$, $B(s) = B$, and $K(s) = sE - A_1 - A_2e^{-s\tau}$.
4) **EM w/ surface loss:** $C(s) = sB$, $B(s) = B$, and $K(s) = s^2M + sL + K - \frac{1}{\sqrt{s}}N$.
5) **Integro-differential Volterra systems, input delays, fractional systems . . .
Given a large-scale SLS

\[ H(s) = C(s)K(s)^{-1}B(s), \]
Given a large-scale SLS

\[ \mathbf{H}(s) = \mathbf{C}(s)\mathbf{K}(s)^{-1}\mathbf{B}(s), \]

find projection matrices

\[ \mathbf{V}, \mathbf{W} \in \mathbb{R}^{n \times r}, \quad \mathbf{W}^T\mathbf{V} = \mathbf{I}_r, \]

(with \( r \ll n \)), such that

\[ \hat{\mathbf{H}}(s) = \hat{\mathbf{C}}(s)\hat{\mathbf{K}}(s)^{-1}\hat{\mathbf{B}}(s), \]

where
Given a large-scale SLS

\[ H(s) = C(s)K(s)^{-1}B(s), \]

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\[ V, W \in \mathbb{R}^{n \times r}, \quad W^T V = I_r, \]

(with \( r \ll n \)), such that

\[ \hat{H}(s) = \hat{C}(s)\hat{K}(s)^{-1}\hat{B}(s), \quad \text{where} \]

\[ \hat{K}(s) = W^T K(s) V, \quad \hat{B}(s) = W^T B(s) \]

and \( \hat{C}(s) = C(s) V \)
Given a large-scale SLS

\[ H(s) = C(s)K(s)^{-1}B(s), \]

find projection matrices

\[ V, W \in \mathbb{R}^{n \times r}, \quad W^T V = I_r, \]

(with \( r \ll n \)), such that

\[ \hat{H}(s) = \hat{C}(s)\hat{K}(s)^{-1}\hat{B}(s), \]

where

\[ \hat{K}(s) = W^T K(s) V, \hat{B}(s) = W^T B(s) \]

and \( \hat{C}(s) = C(s)V \)

- Note \( \hat{A}_i = W^T A_i V, \hat{E} = W^T E V, \hat{C}_i = C_i V \) and \( \hat{B}_i = W^T B_i \).
- The ROM preserves the \( \alpha_i(s), \beta_i(s) \) and \( \gamma_i(s) \) functions.
Interpolation-based methods

- Interpolatory projection methods for structure-preserving model reduction.  
  \[\text{Beattie/Gugercin '09}\]

Interpolation points $\sigma_k, \mu_j \Rightarrow$

\[
\begin{align*}
\mathcal{K}^{-1}(\sigma_k)B(\sigma_k) & \in \text{range (V)} \quad \text{and} \\
\mathcal{K}^{-T}(\mu_k)C^T(\mu_j) & \in \text{range (W)}.
\end{align*}
\]
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- Interpolatory projection methods for structure-preserving model reduction.  
  \[ \text{Beattie/Gugercin '09} \]

Balancing truncation methods

  \[ \text{Breiten '16} \]

\[
\begin{align*}
P &= \frac{1}{2\pi} \int_{-i\infty}^{i\infty} K_s(s)^{-1} \mathcal{B}(s) \mathcal{B}(s)^T K(s)^{-T} ds, \\
Q &= \frac{1}{2\pi} \int_{-i\infty}^{i\infty} K_s(s)^{-T} \mathcal{C}(s)^T \mathcal{C}(s) K(s)^{-1} ds.
\end{align*}
\]

\[ \Rightarrow \text{Find } V, W \text{ from } T^{-1} P Q T = \Sigma. \]
**Interpolation-based methods**

- Interpolatory projection methods for structure-preserving model reduction.
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**Balancing truncation methods**

  \[\text{Breiten '16}\]

**Data-driven methods**

- Data-driven structured realization.
  \[\text{Schulze/Unger/Beattie/Gugercin '18}\]
1. Introduction

2. Minimal Realization
   - Motivation
     - ... of Structured Linear Systems
   - Some Results

3. Reachability and Observability for SLS

4. Model Order Reduction

5. Numerical Results

6. Outlook and Conclusions
Let us consider the first order system

\[ H(s) = C(sI - A)^{-1}B, \quad \text{with} \quad A = \begin{bmatrix} -1 & -1 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \text{and} \quad C^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \]
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Note that

$$H(s) = \frac{1}{s + 2} = \hat{H}(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B},$$

with

$$\hat{A} = -2, \quad \hat{B} = 1 \quad \text{and} \quad \hat{C} = 1.$$
Let us consider the first order system

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Note that \( H(s) = \frac{1}{s+2} = \hat{H}(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B}, \) with \( \hat{A} = -2, \hat{B} = 1 \) and \( \hat{C} = 1. \)

**Minimal realization problem**

Find order \( r \) and matrices \( V \) and \( W \) such that the reduced-order model obtained by projection satisfies

\[ H(s) = \hat{H}(s), \quad \forall s. \]
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### Minimal realization problem

Find order \( r \) and matrices \( V \) and \( W \) such that the reduced-order model obtained by projection satisfies

\[ H(s) = \hat{H}(s), \forall s. \]

### Solutions:

- Kalman reachability/observability criteria,
- Hankel matrix (Silverman method),
- reachability and observability Gramians,
- **Loewner matrix.**  
  [Mayo/Antoulas '07]
For illustration, consider the **time-delay systems**

\[ H(s) = C(sI - A_1 - A_2 e^{-s})^{-1} B, \]

with

\[ A_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \]

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For illustration, consider the **time-delay systems**

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    H(s) = C(sI - A_1 - A_2 e^{-s})^{-1}B, \quad \text{with}
\]

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    A_1 = \begin{bmatrix}
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    \end{bmatrix},
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        1 & 0 & 0 \\
        1 & 0 & 0
    \end{bmatrix},
\]

\[
    B^T = \begin{bmatrix}
        1 & 0 & 0
    \end{bmatrix}
    \quad \text{and}
    \quad C = \begin{bmatrix}
        1 & 1 & 0
    \end{bmatrix}.
\]

\[
    \hat{H}(s) = \hat{C}(sI - \hat{A}_2 - \hat{A}_2 e^{-s})^{-1}\hat{B}, \quad \text{with}
\]

\[
    \hat{A}_1 = \begin{bmatrix}
        -1 & 0 \\
        0 & -1
    \end{bmatrix},
    \hat{A}_2 = \begin{bmatrix}
        1 & 0 \\
        1 & 0
    \end{bmatrix},
\]

\[
    \hat{B} = \begin{bmatrix}
        1 \\
        0
    \end{bmatrix}
    \quad \text{and}
    \quad \hat{C}^T = \begin{bmatrix}
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\[ \hat{H}(s) = \hat{C}(sI - \hat{A}_2 - \hat{A}_2 e^{-s})^{-1} \hat{B}, \quad \text{with} \]

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- \[ H(s) = \hat{H}(s), \forall s. \]
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\( \hat{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and \( \hat{C}^T = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \).

- \( H(s) = \hat{H}(s), \forall s. \)
- \( H \) has order 3 and \( \hat{H} \) order 2.
For illustration, consider the time-delay systems

\[ H(s) = C(sI - A_1 - A_2 e^{-s})^{-1}B, \text{ with } \]

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\[ \hat{H}(s) = \hat{C}(sI - \hat{A}_2 - \hat{A}_2 e^{-s})^{-1}\hat{B}, \text{ with } \]

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Minimal realization problem

Is there a way to find the order \( r \) and matrices \( V, W \in \mathbb{R}^{n \times r} \) such that the system \( \hat{H}(s) \) obtained by projection is "minimal", i.e.

\[ H(s) = \hat{H}(s), \forall s? \]
Given a first order system

\[ H(s) = C(sE - A)^{-1}B, \text{ with } E \in \mathbb{R}^{n \times n} \text{ invertible.} \]
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**Reachability characterization**

[Anderson/Antoulas ’90]

If \((E, A, B)\) is \(\mathbb{R}^n\)-reachable, \(t \geq n\), \(\sigma_i \neq \sigma_j\) for \(i \neq j\), and

\[ R = \begin{bmatrix} (\sigma_1 E - A)^{-1}B & \ldots & (\sigma_t E - A)^{-1}B \end{bmatrix}. \text{ Then } \operatorname{rank}(R) = n. \]
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Observability characterization \([\text{Anderson/Antoulas '90}]\)

If \((E, A, C)\) is \(\mathbb{R}^n\)-observable, \(t \geq n\), \(\sigma_i \neq \sigma_j\) for \(i \neq j\), and

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O = \begin{bmatrix}
(\sigma_1 E - A)^{-T}C^T & \cdots & (\sigma_t E - A)^{-T}C^T
\end{bmatrix}. \text{ Then rank } (O) = n.
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Rank encodes minimality \[\text{[Anderson/Antoulas '90]}\]

\[ \text{rank} \left( O^T E R \right) = \text{order of minimal realization} = r. \]
1. Introduction

2. Minimal Realization

3. Reachability and Observability for SLS
   An Illustrative Example

4. Model Order Reduction

5. Numerical Results

6. Outlook and Conclusions
Reachability and Observability for SLS
Some Results

For **SLS**, we use the notion of $\mathbb{R}^n$ **reachability and observability**. Let us consider the SLS

$$H(s) = C(s)K(s)^{-1}B(s)$$

of order $n$. 

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For \textit{SLS}, we use the notion of $\mathbb{R}^n$ reachability and observability. Let us consider the SLS
\[ H(s) = C(s)K(s)^{-1}B(s) \] of order $n$.

**Reachability characterization**

If $(K(s), B(s))$ is $\mathbb{R}^n$-reachable, $\sigma_i \neq \sigma_j$ for $i \neq j$, $t \geq n$, and
\[
R = [K(\sigma_1)^{-1}B(\sigma_1) \ldots K(\sigma_t)^{-1}B(\sigma_t)],
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then $\text{rank}(R) = n$. 

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### Observability characterization

If $(K(s), B(s))$ is $\mathbb{R}^n$-observable, $\sigma_i \neq \sigma_j$ for $i \neq j$, $t \geq n$, and

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**Rank encodes minimality**

$$\text{rank} \left( O^T ER \right) = \text{order of the SLS } "\text{minimal}" \text{ realization} = r.$$
Let's go back to the time-delay example

\[ H(s) = C(sI - A_1 - A_2 e^{-s})^{-1}B, \]
with

\[ A_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \]

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Let us construct, for \( \sigma_i = [1, 2, 3, 4, 5], \)

\[ R = [K(\sigma_1)^{-1}B \ldots K(\sigma_5)^{-1}B], \quad O = [K(\sigma_1)^{-T}C^T \ldots K(\sigma_5)^{-T}C^T]. \]
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R = \begin{bmatrix} K(\sigma_1)^{-1}B & \ldots & K(\sigma_5)^{-1}B \end{bmatrix}, \quad O = \begin{bmatrix} K(\sigma_1)^{-T}C^T & \ldots & K(\sigma_5)^{-T}C^T \end{bmatrix}.
\]

Hence, we see that

- \( \text{rank} (R) = \text{rank} (O) = 2. \quad (\text{nonreachable nonobservable}) \)
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Hence, we see that

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- \( \text{rank } (O^T R) = 2. \) (minimal realization order)
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- \( \text{rank} (O^T R) = 2 \). (minimal realization order)

Then,

\[ [Y, \Sigma, X] = \text{svd}(O^T R). \]

So, we get the projection matrices

\[ V = RX(:, 1 : 2) \quad \text{and} \quad W = OY(:, 1 : 2). \]
Let’s go back to the time-delay example

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\[
V = RX(:, 1:2) \text{ and } W = OY(:, 1:2).
\]

The \(\hat{H}\) obtained using \(V\) and \(W\) satisfies

\[
H(s) = \hat{H}(s), \forall s.
\]
1. Introduction

2. Minimal Realization

3. Reachability and Observability for SLS

4. Model Order Reduction
   The Basic Approach
   Numerical Implementation
   The Algorithm

5. Numerical Results

6. Outlook and Conclusions
Figure represents the singular values of $O^T ER$ for a large-scale time-delay example.
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For large-scale systems, often low-rank phenomena can be observed.
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- For large-scale systems, often low-rank phenomena can be observed.
- Numerical rank of $O^T ER$ generally small compared to $n$. 
- Figure represents the singular values of $O^T ER$ for a large-scale time-delay example.
- For large-scale systems, often low-rank phenomena can be observed.
- Numerical rank of $O^T ER$ generally small compared to $n$.
- We can cut off states that are related to very small singular value of $O^T ER$. 
To compute $R$ (analogously for $O$),

- we set

$$R_i := \mathcal{K}(\sigma_i)^{-1} \mathcal{B}(\sigma_i), \quad i \in \{1, \ldots, t\}.$$
To compute $\mathbf{R}$ (analogously for $\mathbf{O}$),

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- Hence, if $\mathbf{R} := [R_1, \ldots, R_t]$, it solves

$$\text{ERS} - \sum_{i=1}^{\ell_\alpha} A_i \mathbf{R} \mathbf{M}_i = \sum_{i=1}^m \mathbf{B}_i \mathbf{b}_i,$$

where

$$\mathbf{M}_i = \text{diag} (\alpha_i(\sigma_1), \ldots, \alpha_i(\sigma_t)),$$

$$\mathbf{b}_i = [\beta_i(\sigma_1), \ldots, \beta_i(\sigma_t)],$$

$$\mathbf{S} = \text{diag} (\sigma_1, \ldots, \sigma_t).$$
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- This is a generalized Sylvester equation.
To compute $R$ (analogously for $O$),

- we set

$$R_i := K(\sigma_i)^{-1} B(\sigma_i), \quad i \in \{1, \ldots, t\}.$$ 

- Hence, if $R := [R_1, \ldots, R_t]$, it solves

$$ERS - \sum_{i=1}^{\ell_\alpha} A_i R M_i = \sum_{i=1}^{m} B_i b_i,$$

where

$$M_i = \text{diag}(\alpha_i(\sigma_1), \ldots, \alpha_i(\sigma_t))$$
$$b_i = [\beta_i(\sigma_1), \ldots, \beta_i(\sigma_t)],$$
$$S = \text{diag}(\sigma_1, \ldots, \sigma_t).$$

- This is a generalized Sylvester equation.

- We use the truncated low-rank methods for generalized Sylvester equations from [Kressner/Sirkovic '15].
Algorithm 1 Structure Preserving Numerical Minimal Realization algorithm (SPNMR)

**Input:** SLS $\mathcal{K}(s)$, $\mathcal{B}(s)$, $\mathcal{C}(s)$ and reduced order $r$.

1. Choose interpolation points $(\sigma_1, \ldots, \sigma_t)$.
2. Solve the generalized Sylvester equations for $R$ (and $O$) using a low-rank method.
3. Determine the SVD $[Y, \Sigma, X] = \text{svd}(O^T ER)$.
4. Construct the projection matrices $V = RX(:,1:r)$ and $W = OY(:,1:r)$.

**Output:** Reduced-order model is given by $\hat{\mathcal{K}}(s) = W^T \mathcal{K}(s) V$, $\hat{\mathcal{B}}(s) = W^T \mathcal{B}(s)$ and $\hat{\mathcal{C}}(s) = \mathcal{C}(s) V$. 
Algorithm 1 Structure Preserving Numerical Minimal Realization algorithm (SPNMR)

**Input:** SLS $K(s), B(s), C(s)$ and reduced order $r$.

1. Choose interpolation points $(\sigma_1, \ldots, \sigma_t)$. 

2. Solve the generalized Sylvester equations for $R$ (and $O$) using a low-rank method.

3. Determine the SVD $[Y, \Sigma, X] = \text{svd}(O^T ER)$.

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1. Introduction

2. Minimal Realization

3. Reachability and Observability for SLS

4. Model Order Reduction

5. Numerical Results
   A Time-delay System
   Second-order System
   Parametric Systems
   Fitz-Hugh Nagumo Model

6. Outlook and Conclusions
Let us consider the time delay system

\[
\dot{x}(t) = Ax(t) + A_\tau x(t - \tau) + Bu(t),
\]
\[
y(t) = Cx(t).
\]

- Heated rod cooled using delayed feedback from [Breda/Maset/Vermiglio '09].

- Full order model \( n = 120 \) and \( \tau = 1 \).

- ROM obtained used SPNMR method (100 log. dist. points in \([1e^{-1}, 1e^3]i\)) and Structured Balanced Truncation [Breiten '16].

- Reduced order \( r = 4 \).
Let us consider the time delay system

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- Full order model \( n = 120 \) and \( \tau = 1 \).
- ROM obtained used SPNMR method (100 log. dist. points in \([10^{-1}, 10^3]i\)) and Structured Balanced Truncation [Breiten ’16].
- Reduced order \( r = 12 \).

![Graph showing absolute error](image)
Let us consider the second order system

\[
M \ddot{x}(t) + D \dot{x}(t) + K x(t) = B u(t) \\
y(t) = C x(t).
\]

- Damped vibrational system.
- Full order model with \( n = 301 \).
- ROM obtained using SPNMR method (500 log. dist. points in \([1e^{-3}, 1]\)) and Structured Balanced Truncation [Breiten '16].
- Reduced order \( r = 50 \).
Let us consider the second order system

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The results presented in this talk can also be generalized to parametric SLS, i.e.,

\[ H(s, p) = C(s, p)K(s, p)^{-1}B(s, p). \]
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Consider \( H(s, p) = C(sI - A_1 - pA_2)^{-1}B \), where

\[
A_1 = \begin{bmatrix}
-2 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -2
\end{bmatrix}, \quad A_2 = \begin{bmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix}, \quad \text{and} \quad C^T = \begin{bmatrix}
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$$\mathbf{H}(s, p) = \mathbf{C} (s\mathbf{I} - \mathbf{A}_1 - p\mathbf{A}_2)^{-1} \mathbf{B},$$

where

$$\mathbf{A}_1 = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{C}^T = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$ 

For $t = 20$ points $(\sigma_i, p_i)$, let

$$\mathbf{R} = [K(\sigma_1, p_1)^{-1}\mathbf{B} \ldots K(\sigma_t, p_t)^{-1}\mathbf{B}],$$

$$\mathbf{O} = [K(\sigma_1, p_1)^{-T}\mathbf{C}^T \ldots K(\sigma_t, p_t)^{-T}\mathbf{C}^T].$$
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Build \( O^T R \) and check rank (=2).
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Build \( O^T R \) and check rank (=2).

Compute projectors \( V \) and \( W \) and \( \hat{H}(s, p) \).

Then, \( H(s, p) = \hat{H}(s, p) \).
FOM example [MORwiki]$^1$ of order 1006 and $p \in [10, 100]$ of the form
\[
\dot{x}(t) = (A_1 + pA_2)x(t) + Bu(t)
\]
\[
y(t) = Cx(t)
\]

1500 random points $(s, p) \in [1e0, 1e4] \times [10, 100]$. Reduced order $r = 15$.

---

$^1$morwiki mpi-magdeburg mpg de/
FOM example [MORwiki]¹ of order 1006 and \( p \in [10, 100] \) of the form

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y(t) = Cx(t)
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- 1500 random points \((s, p) \in [\text{1}e0, \text{1}e4]i \times [10, 100]\). Reduced order \( r = 15 \).
- \( p = 10 \)

---

¹morwiki.mpi-magdeburg.mpg.de/
Parametric Systems
Example 2: Parametric FOM

- FOM example [MORWIKI]$^1$ of order 1006 and $p \in [10, 100]$ of the form
  \[
  \dot{x}(t) = (A_1 + pA_2)x(t) + Bu(t) \\
  y(t) = Cx(t)
  \]
- 1500 random points $(s, p) \in [1e0, 1e4]i \times [10, 100]$. Reduced order $r = 15$.
- $p = 55$

---

$^1$morwiki.mpi-magdeburg.mpg.de/
FOM example [MORwiki]\(^1\) of order 1006 and \( p \in [10, 100] \) of the form
\[
\dot{x}(t) = (A_1 + pA_2)x(t) + Bu(t) \\
y(t) = Cx(t)
\]
1500 random points \((s, p) \in [1e0, 1e4]i \times [10, 100]\). Reduced order \( r = 15 \).

\( p = 100 \)

---

\(^1\)morwiki.mpi-magdeburg.mpg.de/
Consider again the FOM model \([\text{MORwiki}]^2\) of order 1006 and \(p \in [10, 100]\) with an artificial delay \((\tau = 3s)\)

\[
\dot{x}(t) = A_1 x(t) + p A_2 x(t - \tau) +Bu(t)
\]

\[
y(t) = Cx(t)
\]

1500 randomly chosen points \((s, p) \in [1e0, 1e4]i \times [10, 100]\). Reduced order \(r = 15\).
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\[
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\]

\[
y(t) = Cx(t)
\]

- 1500 randomly chosen points \((s, p) \in [1e0, 1e4]i \times [10, 100]\). Reduced order \(r = 15\).

- For \(p = 10\), the plots show the magnitude and relative error comparison between the full-order model (FOM) and the reduced-order model (ROM) for different frequencies.
Consider again the FOM model of order 1006 and \( p \in [10, 100] \) with an artificial delay \( (\tau = 3s) \)

\[
\dot{x}(t) = A_1 x(t) + pA_2 x(t - \tau) + Bu(t)
\]

\[
y(t) = Cx(t)
\]

1500 randomly chosen points \((s, p) \in [1e0, 1e4] i \times [10, 100]\). Reduced order \( r = 15 \).

\( p = 55 \)

---

\(^2\text{morwiki.mpi-magdeburg.mpg.de/}\)
Consider again the FOM model \([\text{MORwiki}]^2\) of order 1006 and \(p \in [10, 100]\) with an artificial delay \((\tau = 3s)\)

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\(p = 100\)

\[\text{morwiki.mpi-magdeburg.mpg.de/}\]
Fitz-Hugh Nagumo model: Governing coupled equation

\[ \epsilon v_t = \epsilon^2 v_{xx} + v(v - 0.1)(1 - v) - w + u, \]
\[ w_t = hv - \gamma w + u \]
on \[ [0, T] \times [0, L] \]

with initial and boundary conditions

\[ v(x, 0) = 0, \quad w(x, 0) = 0, \quad x \in (0, L), \quad v_x(0, t) = i_0(t), \quad v_x(L, t) = 0, \quad t \geq 0. \]

- To employ the interpolation-based algorithm, we choose random 100 interpolation points in a logarithmic way between \([10^{-2}, 10^2]\) and set \(\sigma_i = \mu_i, i \in \{1, \ldots, 100\}\).
Fitz-Hugh Nagumo model: Governing coupled equation

\[
\epsilon v_t = \epsilon^2 v_{xx} + v(v - 0.1)(1 - v) - w + u, \quad \text{on} \quad [0, T] \times [0, L]
\]

\[
w_t = hw - \gamma w + u
\]

Decay of singular values of Loewner pencil

\[
\text{svd}\left([L, L_s]\right)
\]
Fitz-Hugh Nagumo model: Governing coupled equation

\[ \epsilon v_t = \epsilon^2 v_{xx} + v(v - 0.1)(1 - v) - w + u, \]
\[ w_t = hv - \gamma w + u \]
on \[ [0, T] \times [0, L] \]

Construction of reduced systems

- Ori. sys. \((n = 300)\)
- Red. sys. \((r = 15)\)
- Red. sys. \((r = 6)\)
1. Introduction

2. Minimal Realization

3. Reachability and Observability for SLS

4. Model Order Reduction

5. Numerical Results

6. Outlook and Conclusions
Contribution of this talk

- Minimal realization by projection of SLS.
- Model reduction technique inspired by numerical rank of matrix $O^T E R$.
- Projector computation solving generalized Sylvester equation (low-rank methods).
- Performance illustrated by numerical examples for several system classes.
- Extended results to parametric SLS.
Outlook and Conclusions

Contribution of this talk

- Minimal realization by projection of SLS.
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- Performance illustrated by numerical examples for several system classes.
- Extended results to parametric SLS.

Open questions and future work

- Stability preservation and error bounds.
- Relation to pure Loewner-style approach [Schulze/Unger/Beattie/Gugercin ’18]?
- Extension to nonlinear systems, first results for polynomial systems in [Benner/Goyal ’19, arXiv:1904.11891].