NUMERICAL SOLUTION OF LARGE-SCALE ALGEBRAIC RICCATI EQUATIONS

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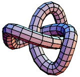

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Overview

1 Solving Large-Scale AREs
- Motivation
- Basic approaches
- Low-Rank Approximation
- Low-Rank Krylov Subspace Methods

2 Newton's Method for AREs
- ADI Method for Lyapunov Equations
- Low-Rank Newton-ADI for AREs
- Application to LQR Problem
- Numerical Results

3 AREs with Indefinite Hessian
- H_∞-Control
- Lyapunov Iterations/Perturbed Hessian Approach
- Riccati Iterations
- Numerical example

4 Software

5 Conclusions and Open Problems

6 References

Solving Large-Scale AREs

Algebraic Riccati Equation (ARE)

General form for $A, G = G^T, W = W^T \in \mathbb{R}^{n\times n}$ given and $X \in \mathbb{R}^{n\times n}$ unknown:

$$0 = \mathcal{R}(X) := A^T X + XA - XGX + W.$$

Large-scale AREs from semi-discretized PDE control problems:

- $n = 10^3 - 10^6$ ($\Longrightarrow 10^6 - 10^{12}$ unknowns!),
- A has sparse representation ($A = -M^{-1}L$ for FEM),
- G, W low-rank with $G, W \in \{BB^T, C^T C\}$, where $B \in \mathbb{R}^{n\times m}$, $m \ll n$, $C \in \mathbb{R}^{p\times n}$, $p \ll n$.
- Standard (eigenproblem-based) $\mathcal{O}(n^3)$ methods are not applicable!

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Large-scale AREs
Peter Benner
Solving Large-Scale AREs
Motivation
Basic approaches
Low-Rank Approximation
Low-Rank Krylov Subspace Methods
Newton's Method for AREs
AREs with Indefinite Hessian
Software
Conclusions and Open Problems
References

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Solving Large-Scale AREs
Motivation
Basic approaches
Low-Rank Approximation
Low-Rank Krylov Subspace Methods

Newton's Method for AREs

AREs with Indefinite Hessian

Software

Conclusions and Open Problems

References

Solving Large-Scale AREs

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Large-scale AREs

Peter Benner

Solving Large-Scale AREs
Motivation
Basic approaches
Low-Rank Approximation
Low-Rank Krylov Subspace Methods

Newton's Method for AREs

AREs with Indefinite Hessian

Software

Conclusions and Open Problems

References

Numerical solution of linear-quadratic optimal control problem for parabolic PDEs via Galerkin approach, spatial FEM discretization $\rightsquigarrow$

LQR Problem (finite-dimensional)

$$\text{Min } \mathcal{J}(u) = \frac{1}{2}\int_0^\infty \left(y^T Q y + u^T R u\right) dt \quad \text{for} \quad u \in \mathcal{L}_2(0,\infty;\mathbb{R}^m),$$

subject to $M\dot{x} = -Lx + Bu, \quad x(0) = x_0, \quad y = Cx,$
with stiffness $L \in \mathbb{R}^{n\times n}$, mass $M \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$.

Solution of finite-dimensional LQR problem: feedback control

$$u_*(t) = -B^T X_* x(t) =: -K_* x(t),$$

where $X_* = X_*^T \geq 0$ is the unique stabilizing[1] solution of the ARE

$$0 = \mathcal{R}(X) := C^T C + A^T X + XA - XBB^T X,$$

with $A := -M^{-1}L$, $B := M^{-1}BR^{-\frac{1}{2}}$, $C := CQ^{-\frac{1}{2}}$.

[1] X is stabilizing $\Leftrightarrow \Lambda(A - BB^T X) \subset \mathbb{C}^-$.

Large-scale AREs

Peter Benner

Solving Large-Scale AREs
Motivation
Basic approaches
Low-Rank Approximation
Low-Rank Krylov Subspace Methods

Newton's Method for AREs

AREs with Indefinite Hessian

Software

Conclusions and Open Problems

References

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[1] X is stabilizing $\Leftrightarrow \Lambda(A - BB^TX) \subset \mathbb{C}^-$.

Treat ARE as nonlinear system of equations

Use Newton's method and its relatives, requires efficient solution of one Lyapunov equation

$$(A - GX_j)^T N_j + N_j(A - GX_j) = -\mathcal{R}(X_j)$$

per iteration. Using Hammarling's method or sign function iteration, this requires $\mathcal{O}(n^3)$ operations.

Sign function becomes feasible using hierarchical matrix structure and formatted arithmetic [Hackbusch/Khoromskij/Grasedyck 2003, Baur/B. 2006].

Large-scale AREs

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Solving Large-Scale AREs
- Motivation
- Basic approaches
- Low-Rank Approximation
- Low-Rank Krylov Subspace Methods

Newton's Method for AREs

AREs with Indefinite Hessian

Software

Conclusions and Open Problems

References

Treat ARE as nonlinear system of equations

Solve corresponding Hamiltonian eigenproblem

Let

$$\underbrace{\begin{bmatrix} A & G \\ W & -A^T \end{bmatrix}}_{=:H} \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} U \\ V \end{bmatrix} T,$$

so that colspan$\left(\begin{bmatrix} U \\ V \end{bmatrix}\right) = \mathcal{U} \in \mathbb{R}^{2n \times n}$ is the maximal stable H-invariant subspace (i.e., $\Lambda(T) \subset \mathbb{C}^-$, $\dim \mathcal{U} = n$), then (under mild assumptions),

$$X_* = -VU^{-1}$$

is the stabilizing solution of the ARE.

Can be computed using Schur decomposition of H and related techniques, but: complexity of all methods is $\mathcal{O}(n^3)$!

Several attempts to use partial Schur decomposition and Krylov subspace methods $\rightsquigarrow$ later.

Large-scale AREs
Peter Benner
Solving Large-Scale AREs
Motivation
Basic approaches
Low-Rank Approximation
Low-Rank Krylov Subspace Methods
Newton's Method for AREs
AREs with Indefinite Hessian
Software
Conclusions and Open Problems
References

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Solving Large-Scale AREs
- Motivation
- Basic approaches
- Low-Rank Approximation
- Low-Rank Krylov Subspace Methods

Newton's Method for AREs

AREs with Indefinite Hessian

Software

Conclusions and Open Problems

References

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Low-Rank Approximation

ARE $0 = A^T X + XA - XBB^T X + CC^T$

Large-scale AREs

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Solving Large-Scale AREs
- Motivation
- Basic approaches
- Low-Rank Approximation
- Low-Rank Krylov Subspace Methods

Newton's Method for AREs

AREs with Indefinite Hessian

Software

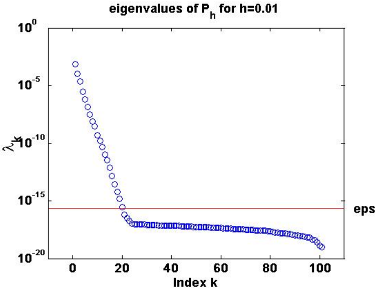
Conclusions and Open Problems

References

Consider spectrum of ARE solution:

Example:

- Linear 1D heat equation with point control,
- $\Omega = [0, 1]$,
- FEM discretization using linear B-splines,
- $h = 1/100 \implies n = 101$.



Idea: $X = X^T \geq 0 \implies$

$$X = YY^T = \sum_{k=1}^{n} \lambda_k y_k y_k^T \approx \sum_{k=1}^{r} \lambda_k y_k y_k^T =: Y^{(r)}(Y^{(r)})^T.$$

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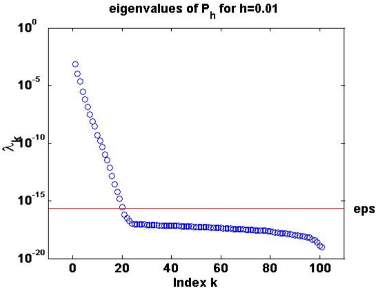

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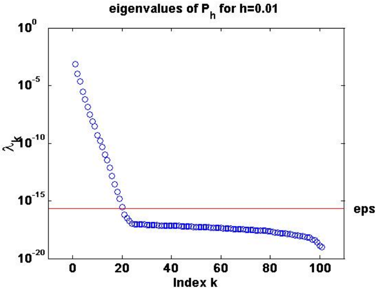

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Large-scale AREs
Peter Benner
Solving Large-Scale AREs
Motivation
Basic approaches
Low-Rank Approximation
Low-Rank Krylov Subspace Methods
Newton's Method for AREs
AREs with Indefinite Hessian
Software
Conclusions and Open Problems
References

Consider $0 = \mathcal{R}(X) = CC^T + AX + XA^T - XBB^TX.$

1. Apply (block-)Arnoldi process to A with start (block-)vector C to generate the Krylov space

$$\mathcal{K}_\ell(A, C) = \operatorname{span}\{C, AC, A^2C, \dots, A^{\ell-1}C\}$$

with orthogonal basis V_ℓ such that

$$AV_\ell = V_\ell A_\ell + W_{\ell+1}A_{\ell+1,\ell}\begin{bmatrix} 0 & I_p \end{bmatrix}$$

and $A_\ell = V_\ell^T A V_\ell$ is block upper-Hessenberg.

2. Set $B_\ell := V_\ell^T B$, $C_\ell := V_\ell^T C$.
3. Find stabilizing solution of the ARE

$$0 = \mathcal{R}_\ell(X_\ell) = C_\ell C_\ell^T + A_\ell X_\ell + X_\ell A_\ell^T - X_\ell B_\ell B_\ell^T X_\ell.$$

4. Set $\tilde{X} := V_\ell X_\ell V_\ell^T$.

Low-Rank Krylov Subspace Methods

Block-Arnoldi method

[Jaimoukha/Kasenally '94]

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Block-Arnoldi method [Jaimoukha/Kasenally '94]

Consider $0 = \mathcal{R}(X) = CC^T + AX + XA^T - XBB^TX.$

Properties:

+ $\tilde{X}$ satisfies Galerkin-type condition $V_\ell^T \mathcal{R}(\tilde{X}) V_\ell = 0.$
+ Computable residual error norm

$$\|\mathcal{R}(\tilde{X})\|_F = \sqrt{2} \cdot \|A_{\ell+1,\ell} \begin{bmatrix} 0 & I_p \end{bmatrix} X_\ell\|_F.$$

– Block-Arnoldi, i.e., each step needs p matrix-vector products.
– Stabilizing X_ℓ may not exist as corresponding Hamiltonian matrix

$$H_\ell := \begin{bmatrix} A_\ell^T & B_\ell B_\ell^T \\ C_\ell C_\ell^T & -A_\ell \end{bmatrix}$$

may have purely imaginary eigenvalues!
– No stabilization guarantee for $\tilde{X}$!
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Large-scale AREs
Peter Benner
Solving Large-Scale AREs
Motivation
Basic approaches
Low-Rank Approximation
Low-Rank Krylov Subspace Methods
Newton's Method for AREs
AREs with Indefinite Hessian
Software
Conclusions and Open Problems
References

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$$0 = \mathcal{R}(X) = C^T C + A^T X + XA - XBB^T X.$$

1. Apply symplectic Lanczos method to Hamiltonian matrix $\begin{bmatrix} A & BB^T \\ C^T C & -A^T \end{bmatrix}$ to generate Krylov subspace

$$\mathcal{K}_{2\ell}(H, v_1) = \text{span}\{v_1, Hv_1, H^2 v_1, \ldots, H^{2\ell-1} v_1\},$$

with symplectic basis $S_\ell = [v_1, w_1, \ldots, v_\ell, w_\ell] \in \mathbb{R}^{2n,2\ell}$, i.e.,

$$S_\ell^T \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} S_\ell = \begin{bmatrix} 0 & I_\ell \\ -I_\ell & 0 \end{bmatrix}$$

such that

$$HS_\ell = S_\ell H_\ell + \zeta_{\ell+1} v_{\ell+1} e_{2\ell}^T.$$

2. Compute ARE solution corresponding to H_ℓ, prolongate to $\mathbb{R}^{n \times n}$ to obtain approximate rank-ℓ solution $\tilde{X}$.

Large-scale AREs
Peter Benner
Solving Large-Scale AREs
Motivation
Basic approaches
Low-Rank Approximation
Low-Rank Krylov Subspace Methods
Newton's Method for AREs
AREs with Indefinite Hessian
Software
Conclusions and Open Problems
References

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Properties

+ $\tilde{X}$ satisfies Galerkin-type condition $\quad Z_\ell^T \mathcal{R}(\tilde{X}) Z_\ell = 0.$
+ In general, less (but twice as expensive) matrix-vector products than for block-Arnoldi.
+ Purely imaginary eigenvalues of small Hamiltonian matrix H_ℓ can be removed by implicit restarts, i.e., can always get stable H_ℓ-invariant subspace.
+ Stabilization property for projected feedback matrix

$$Z_\ell^T (A - BB^T \tilde{X}) Z_\ell.$$

for sufficiently small Lanczos residual (can be achieved by implicit restarts).

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– No stabilization guarantee for $\tilde{X}$.
– No convergence results for residuals.

Properties

+ $\tilde{X}$ satisfies Galerkin-type condition $\quad Z_\ell^T \mathcal{R}(\tilde{X}) Z_\ell = 0.$
+ In general, less (but twice as expensive) matrix-vector products than for block-Arnoldi.
+ Purely imaginary eigenvalues of small Hamiltonian matrix H_ℓ can be removed by implicit restarts, i.e., can always get stable H_ℓ-invariant subspace.
+ Stabilization property for projected feedback matrix

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Related work

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  [B./Mehrmann/Kressner/Xu in progress]

- Improved generation of low-rank approximate solution from r-dimensional ($r \ll n$) H-invariant subspace $\mathrm{range}(W)$ via

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\begin{aligned}
&W_1 := W(1:n,:), \qquad W_2 = W(n+1:2n,:)\\
&[U,\Sigma,V] = \mathrm{svd}(W_1^T W_2),\\
&\tilde{X} = W_2 U S^{-\frac{1}{2}} \quad (= \tilde{X}^T).
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  [Amodei '03, B./Mehrmann/Sorensen '03]

- Variants of Arnoldi method using Frobenius inner product $\rightsquigarrow$ global Arnoldi. [Jbilou '03–'06]

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Newton's Method for AREs

[Kleinman '68, Mehrmann '91, Lancaster/Rodman '95, B./Byers '94/'98, B. '97, Guo/Laub '99]

- Consider $0 = \mathcal{R}(X) = C^T C + A^T X + XA - XBB^T X$.
- Frechét derivative of $\mathcal{R}(X)$ at X:

$$\mathcal{R}'_X : Z \to (A - BB^T X)^T Z + Z(A - BB^T X).$$

- Newton-Kantorovich method:

$$X_{j+1} = X_j - \left(\mathcal{R}'_{X_j}\right)^{-1} \mathcal{R}(X_j), \quad j = 0, 1, 2, \ldots$$

Newton's method (with line search) for AREs

FOR $j = 0, 1, \ldots$

1. $A_j \leftarrow A - BB^T X_j =: A - BK_j$.
2. Solve the Lyapunov equation $A_j^T N_j + N_j A_j = -\mathcal{R}(X_j)$.
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  - $A_j = A - BK_j = A - BB^T X_j$ is stable $\forall\, j \geq 0$.
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- Need large-scale Lyapunov solver; here, ADI iteration: linear systems with dense, but "sparse+low rank" coefficient matrix A_j:

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$$=\; \text{sparse} \;-\; m \;\cdot\; \square$$

- $m \ll n \implies$ efficient "inversion" using Sherman-Morrison-Woodbury formula:

$$(A - BK_j)^{-1} = (I_n + A^{-1}B(I_m - K_jA^{-1}B)^{-1}K_j)A^{-1}.$$

- BUT: $X = X^T \in \mathbb{R}^{n\times n} \implies n(n+1)/2$ unknowns!

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Low-Rank Krylov Subspace Methods

Properties and Implementation

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ADI Method for Lyapunov Equations

- For $A \in \mathbb{R}^{n \times n}$ stable, $B \in \mathbb{R}^{n \times m}$ ($w \ll n$), consider Lyapunov equation
$$AX + XA^T = -BB^T.$$
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$$\begin{aligned}(A + p_k I)X_{(k-1)/2} &= -BB^T - X_{k-1}(A^T - p_k I)\\ (A + \overline{p_k} I){X_k}^T &= -BB^T - X_{(k-1)/2}(A^T - \overline{p_k} I)\end{aligned}$$
with parameters $p_k \in \mathbb{C}^-$ and $p_{k+1} = \overline{p_k}$ if $p_k \notin \mathbb{R}$.
- For $X_0 = 0$ and proper choice of p_k: $\lim_{k \to \infty} X_k = X$ superlinear.
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Factored ADI Iteration

Lyapunov equation $0 = AX + XA^T = -BB^T$.

Setting $X_k = Y_k Y_k^T$, some algebraic manipulations $\Longrightarrow$

Algorithm [Penzl 1997, Li/White 2002, B./Li/Penzl 1999/2008]

$$V_1 \leftarrow \sqrt{-2\mathrm{Re}\,(p_1)}(A + p_1 I)^{-1} B, \qquad Y_1 \leftarrow V_1$$

FOR $j = 2, 3, \ldots$

$$V_k \leftarrow \sqrt{\frac{\mathrm{Re}\,(p_k)}{\mathrm{Re}\,(p_{k-1})}} \left(V_{k-1} - (p_k + \overline{p_{k-1}})(A + p_k I)^{-1} V_{k-1}\right),$$

$$Y_k \leftarrow \begin{bmatrix} Y_{k-1} & V_k \end{bmatrix}$$

$$Y_k \leftarrow \mathrm{rrqr}(Y_k, \tau) \qquad \text{\% column compression}$$

At convergence, $Y_{k_{\max}} Y_{k_{\max}}^T \approx X$, where

$$\mathrm{range}\,(Y_{k_{\max}}) = \mathrm{range}\left(\begin{bmatrix} V_1 & \ldots & V_{k_{\max}} \end{bmatrix}\right), \quad V_k = \begin{bmatrix} \, \end{bmatrix} \in \mathbb{C}^{n \times m}.$$

Note: Implementation in real arithmetic possible by combining two steps.

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Low-Rank Newton-ADI for AREs

Re-write Newton's method for AREs

$$A_j^T N_j + N_j A_j = -\mathcal{R}(X_j)$$

$$\Longleftrightarrow$$

$$A_j^T \underbrace{(X_j + N_j)}_{=X_{j+1}} + \underbrace{(X_j + N_j)}_{=X_{j+1}} A_j = \underbrace{-C^T C - X_j B B^T X_j}_{=:-W_j W_j^T}$$

Set $X_j = Z_j Z_j^T$ for $\mathrm{rank}\,(Z_j) \ll n \Longrightarrow$

$$A_j^T \left(Z_{j+1} Z_{j+1}^T\right) + \left(Z_{j+1} Z_{j+1}^T\right) A_j = -W_j W_j^T$$

Factored Newton Iteration [B./Li/Penzl 1999/2008]

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Optimal feedback

$$K_* = B^T X_* = B^T Z_* Z_*^T$$

can be computed by direct feedback iteration:

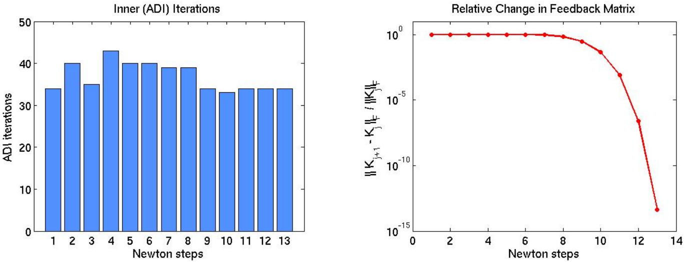
- jth Newton iteration:

$$K_j = B^T Z_j Z_j^T = \sum_{k=1}^{k_{\max}} (B^T V_{j,k}) V_{j,k}^T \quad \xrightarrow{j\to\infty} \quad K_* = B^T Z_* Z_*^T$$

- K_j can be updated in ADI iteration, no need to even form Z_j, need only fixed workspace for $K_j \in \mathbb{R}^{m\times n}$!

Related to earlier work by [Banks/Ito 1991].

- Linear 2D heat equation with homogeneous Dirichlet boundary and point control/observation.
- FD discretization on uniform 150×150 grid.
- $n = 22.500$, $m = p = 1$, 10 shifts for ADI iterations.
- Convergence of large-scale matrix equation solvers:



Performance of Newton's method for accuracy $\sim 1/n$

grid	unknowns	$\frac{\|\mathcal{R}(X)\|_F}{\|X\|_F}$	it. (ADI it.)	CPU (sec.)
8×8	2,080	4.7e-7	2 (8)	0.47
16×16	32,896	1.6e-6	2 (10)	0.49
32×32	524,800	1.8e-5	2 (11)	0.91
64×64	8,390,656	1.8e-5	3 (14)	7.98
128×128	134,225,920	3.7e-6	3 (19)	79.46

Here,

- Convection-diffusion equation,
- $m = 1$ input and $p = 2$ outputs,
- $X = X^T \in \mathbb{R}^{n \times n} \Rightarrow \frac{n(n+1)}{2}$ unknowns.

Confirms mesh independence principle for Newton-Kleinman [Burns/Sachs/Zietsmann 2006].

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Now:

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Problems

- For large-scale problems, resulting, e.g., from H_∞ control, standard methods based on Hamiltonian/even eigenvalue problem can not be used due to $\mathcal{O}(n^3)$ complexity/dense matrix algebra.
- Krylov subspace methods might be employed, but so far no convergence results, and in case of convergence, no guarantee that stabilizing solution is computed.
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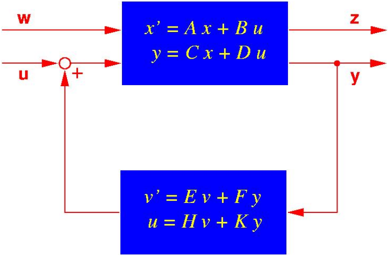
Motivation: H_∞-Control

Linear time-invariant systems

$$\Sigma : \begin{cases} \dot{x} = Ax + B_1 w + B_2 u, \\ z = C_1 x + D_{11} w + D_{12} u, \\ y = C_2 x + D_{21} w + D_{22} u, \end{cases}$$

where $A \in \mathbb{R}^{n \times n}$, $B_k \in \mathbb{R}^{n \times m_k}$, $C_j \in \mathbb{C}^{p_j \times n}$, $D_{jk} \in \mathbb{R}^{p_j \times m_k}$.

x – states of the system,
w – exogenous inputs
u – control inputs,
z – performance outputs
y – measured outputs



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Conclusions and Open Problems
References

Laplace transform $\Longrightarrow$ transfer function (in frequency domain)

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \equiv \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right].$$

where for $x(0) = 0$, G_{ij} are the rational matrix functions

- $G_{11}(s) = C_1(sI - A)^{-1}B_1 + D_{11}$,
- $G_{12}(s) = C_1(sI - A)^{-1}B_2 + D_{12}$,
- $G_{21}(s) = C_2(sI - A)^{-1}B_1 + D_{21}$,
- $G_{22}(s) = C_2(sI - A)^{-1}B_2 + D_{22}$,

describing the transfer from inputs to outputs of Σ via

$$\begin{aligned} z(s) &= G_{11}(s)w(s) + G_{12}(s)u(s), \\ y(s) &= G_{21}(s)w(s) + G_{22}(s)u(s). \end{aligned}$$

Large-scale AREs

Peter Benner

Solving Large-Scale AREs

Newton's Method for AREs

AREs with Indefinite Hessian

H_∞-Control

Lyapunov Iterations/Perturbed Hessian Approach

Riccati Iterations

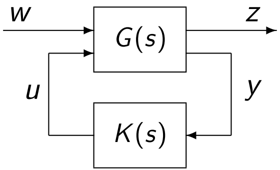
Numerical example

Software

Conclusions and Open Problems

References

Consider closed-loop system, where $K(s)$ is an internally stabilizing controller, i.e., K stabilizes G for $w \equiv 0$.



Large-scale AREs
Peter Benner
Solving Large-Scale AREs
Newton's Method for AREs
AREs with Indefinite Hessian
H_∞-Control
Lyapunov Iterations/Perturbed Hessian Approach
Riccati Iterations
Numerical example
Software
Conclusions and Open Problems
References

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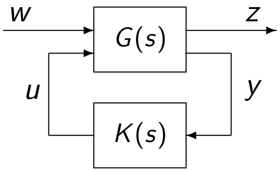

Goal:

find K that minimize error outputs

$$z = \left(G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}\right) w =: \mathcal{F}(G, K)w,$$

where $\mathcal{F}(G, K)$ is the linear fractional transformation of G, K.

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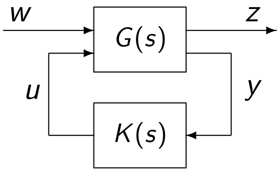

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H_∞-optimal control problem:

$$\min_{K \text{ stabilizing}} \|\mathcal{F}(G, K)\|_{\mathcal{H}_\infty}.$$

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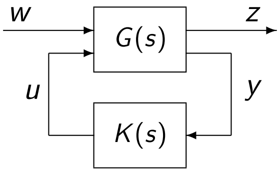

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where $\mathcal{F}(G,K)$ is the linear fractional transformation of G, K.

H_∞-suboptimal control problem:

For given constant $\gamma > 0$, find all internally stabilizing controllers satisfying

$$\|\mathcal{F}(G,K)\|_{\mathcal{H}_\infty} < \gamma.$$

Simplifying assumptions

1. $D_{11} = 0$;
2. $D_{22} = 0$;
3. (A, B_1) stabilizable, (C_1, A) detectable;
4. (A, B_2) stabilizable, (C_2, A) detectable ($\Longrightarrow \Sigma$ internally stabilizable);
5. $D_{12}^T [\, C_1 \;\; D_{12} \,] = [\, 0 \;\; I_{m_2} \,]$;
6. $\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}^T = \begin{bmatrix} 0 \\ I_{p_2} \end{bmatrix}$.

Remark. 1.,2.,5.,6. only for notational convenience, 3. can be relaxed, but derivations get even more complicated.

Theorem [Doyle/Glover/Khargonekar/Francis '89]

Given the Assumptions 1.–6., there exists an admissible controller $K(s)$ solving the H_∞-suboptimal control problem $\Longleftrightarrow$

(i) There exists a solution $X_\infty = X_\infty^T \geq 0$ to the ARE

$$C_1C_1^T + A^TX + XA + X(\gamma^{-2}B_1B_1^T - B_2B_2^T)X = 0, \qquad (1)$$

such that A_X is Hurwitz, where $A_X := A + (\gamma^{-2}B_1B_1^T - B_2B_2^T)X_\infty$.

(ii) There exists a solution $Y_\infty = Y_\infty^T \geq 0$ to the ARE

$$B_1B_1^T + AY + YA^T + Y(\gamma^{-2}C_1C_1^T - C_2C_2^T)Y = 0, \qquad (2)$$

such that A_Y is Hurwitz where $A_Y := A + Y_\infty(\gamma^{-2}C_1C_1^T - C_2C_2^T)$.

(iii) $\gamma^2 > \rho(X_\infty Y_\infty)$.

H_∞-optimal control

Find minimal γ for which (i)–(iii) are satisfied $\rightsquigarrow$ γ-iteration based on solving AREs (1)–(2) repeatedly for different γ.

Large-scale AREs

Peter Benner

Solving Large-Scale AREs

Newton's Method for AREs

AREs with Indefinite Hessian

H_∞-Control

Lyapunov Iterations/Perturbed Hessian Approach

Riccati Iterations

Numerical example

Software

Conclusions and Open Problems

References

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(ii) There exists a solution $Y_\infty = Y_\infty^T \geq 0$ to the ARE

$$B_1B_1^T + AY + YA^T + Y(\gamma^{-2}C_1C_1^T - C_2C_2^T)Y = 0, \tag{2}$$

such that A_Y is Hurwitz where $A_Y := A + Y_\infty(\gamma^{-2}C_1C_1^T - C_2C_2^T)$.

(iii) $\gamma^2 > \rho(X_\infty Y_\infty)$.

H_∞-optimal control

Find minimal γ for which (i)–(iii) are satisfied $\rightsquigarrow$ γ-iteration based on solving AREs (1)–(2) repeatedly for different γ.

H_∞-(sub-)optimal controller

If (i)–(iii) hold, a suboptimal controller is given by

$$\hat{K}(s) = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & 0 \end{array}\right] = \hat{C}(sI_n - \hat{A})^{-1}\hat{B},$$

where for

$$Z_\infty := (I - \gamma^{-2} Y_\infty X_\infty)^{-1},$$

$$\begin{aligned} \hat{A} &:= A + (\gamma^{-2} B_1 B_1^T - B_2 B_2^T) X_\infty - Z_\infty Y_\infty C_2^T C_2, \\ \hat{B} &:= Z_\infty Y_\infty C_2^T, \\ \hat{C} &:= -B_2^T X_\infty. \end{aligned}$$

$\hat{K}(s)$ is the central or minimum entropy controller.

Numerical Solution of AREs with Indefinite Hessian

A quick-and-dirty solution [DAMM 2002/04]

ARE with indefinite Hessian

$$0 = \mathcal{R}(X) := C^T C + A^T X + XA + X(B_1 B_1^T - B_2 B_2^T)$$

Consider $X^{-1}\mathcal{R}(X)X^{-1} = 0$

$\rightsquigarrow$ standard ARE for $\tilde{X} \equiv X^{-1}$

$$\tilde{\mathcal{R}}(\tilde{X}) := (B_1 B_1^T - B_2 B_2^T) + \tilde{X} A^T + A\tilde{X} + \tilde{X} C^T C \tilde{X} = 0.$$

Newton's method will converge to stabilizing solution, Newton-ADI can be employed (with modification for indefinite constant term).

But: low-rank approximation of $\tilde{X}$ will not yield good approximation of $X \Rightarrow$ not feasible for large-scale problems!

Lyapunov Iterations/Perturbed Hessian Approach

[Cherfi/Abou-Kandil/Bourles 2005 (Proc. ACSE 2005)]

Idea

Perturb Hessian to enforce semi-definiteness: write

$$0 = A^T X + XA + Q - XGX = A^T X + XA + Q - XDX + X(D - G)X,$$

where $D = G + \alpha I \geq 0$ with $\alpha \geq \min\{0, -\lambda_{\max}(G)\}$.

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Here: $G = B_2 B_2^T - B_1 B_1^T$

$\Rightarrow$ use $\alpha = \|B_1\|^2$ for spectral/Frobenius norm or

$$\alpha = \|B_1\|_1 \cdot \|B_1\|_\infty.$$

Remark

$W \geq -G$ can be used instead of αI, e.g., $W = \beta B_1 B_1^T$ with $\beta \geq 1$.

Large-scale AREs
Peter Benner
Solving Large-Scale AREs
Newton's Method for AREs
AREs with Indefinite Hessian
H_∞-Control
Lyapunov Iterations/Perturbed Hessian Approach
Riccati Iterations
Numerical example
Software
Conclusions and Open Problems
References

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Lyapunov iteration

Based on

$$(A - DX)^T X + X(A - DX) = -Q - XDX - \alpha X^2,$$

iterate

FOR $k = 0, 1, \ldots$, solve Lyapunov equation

$$(A - DX_k)^T X_{k+1} + X_{k+1}(A - DX_k) = -Q - X_k D X_k - \alpha X_k^2.$$

Lyapunov Iterations/Perturbed Hessian Approach

[Cherfi/Abou-Kandil/Bourles 2005 (Proc. ACSE 2005)]

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Easy to convert to low-rank iteration employing low-rank ADI for Lyapunov equations, e.g. with $W = B_1 B_1^T$ instead of αI: the Lyapunov equation becomes

$$\begin{aligned}(A - B_2 B_2^T Y_k Y_k)^T Y_{k+1} Y_{k+1}^T + Y_{k+1} Y_{k+1}^T (A - B_2 B_2^T Y_k Y_k) \\ = \quad -CC^T - Y_k Y_k^T B_1 B_1^T Y_k Y_k^T - Y_k Y_k^T B_2 B_2^T Y_k Y_k^T \\ = \quad -[\, C,\ Y_k Y_k^T B_1,\ Y_k Y_k^T B_2 \,] \begin{bmatrix} C^T \\ B_1^T Y_k Y_k^T \\ B_2^T Y_k Y_k^T \end{bmatrix}.\end{aligned}$$

Large-scale AREs
Peter Benner
Solving Large-Scale AREs
Newton's Method for AREs
AREs with Indefinite Hessian
H_∞-Control
Lyapunov Iterations/Perturbed Hessian Approach
Riccati Iterations
Numerical example
Software
Conclusions and Open Problems
References

Theorem [Cherfi/Abou-Kandil/Bourles 2005]

If

- $\exists\ \hat{X}$ such that $\mathcal{R}(\hat{X}) \geq 0$,
- $\exists\ X_0 = X_0^T \geq \hat{X}$ such that $\mathcal{R}(X_0) \leq 0$ and $A - DX_0$ is Hurwitz,

then

a) $X_0 \geq \ldots \geq X_k \geq X_{k+1} \geq \ldots \geq \hat{X}$,

b) $\mathcal{R}(X_k) \leq 0$ for all $k = 0, 1, \ldots$,

c) $A - DX_k$ is Hurwitz for all $k = 0, 1, \ldots$,

d) $\exists \quad \lim_{k\to\infty} X_k =: \underline{X} \geq \hat{X}$,

e) $\underline{X}$ is semi-stabilizing.

Main problems

- Conditions for initial guess make its computation difficult.
- Observed convergence is linear.

Large-scale AREs
Peter Benner
Solving Large-Scale AREs
Newton's Method for AREs
AREs with Indefinite Hessian
H_∞-Control
Lyapunov Iterations/Perturbed Hessian Approach
Riccati Iterations
Numerical example
Software
Conclusions and Open Problems
References

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Large-scale AREs
Peter Benner
Solving Large-Scale AREs
Newton's Method for AREs
AREs with Indefinite Hessian
H_∞-Control
Lyapunov Iterations/Perturbed Hessian Approach
Riccati Iterations
Numerical example
Software
Conclusions and Open Problems
References

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Large-scale AREs

Peter Benner

Solving Large-Scale AREs

Newton's Method for AREs

AREs with Indefinite Hessian

H_∞-Control

Lyapunov Iterations/Perturbed Hessian Approach

Riccati Iterations

Numerical example

Software

Conclusions and Open Problems

References

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Riccati Iterations

[Lanzon/Feng/B.D.O. Anderson 2007 (Proc. ECC 2007)]

Idea

Consider

$$A^T X + XA + C^T C + X(B_1 B_1^T - B_2 B_2^T)X =: \mathcal{R}(X).$$

Then

$$\mathcal{R}(X+Z) = \mathcal{R}(X) + (\underbrace{A + (B_1 B_1^T - B_2 B_2^T)X}_{=:\widehat{A}})^T Z + Z\widehat{A} + Z(B_1 B_1^T - B_2 B_2^T)Z.$$

Furthermore, if $X = X^T$, $Z = Z^T$ solve the standard ARE

$$0 = \mathcal{R}(X) + \widehat{A}^T Z + Z\widehat{A} - ZB_2 B_2^T Z,$$

then

$$\begin{aligned} \mathcal{R}(X+Z) &= ZB_1 B_1^T Z, \\ \|\mathcal{R}(X)\|_2 &= \|B_1^T Z\|_2. \end{aligned}$$

Large-scale AREs

Peter Benner

Solving Large-Scale AREs

Newton's Method for AREs

AREs with Indefinite Hessian

H_∞-Control

Lyapunov Iterations/Perturbed Hessian Approach

Riccati Iterations

Numerical example

Software

Conclusions and Open Problems

References

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$$\|\mathcal{R}(X)\|_2 = \|B_1^T Z\|_2.$$

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Riccati Iterations

[Lanzon/Feng/B.D.O. Anderson 2007 (Proc. ECC 2007)]

Riccati iteration

1. Set $X_0 = 0$.
2. FOR $k = 1, 2, \ldots,$
   - (i) Set $A_k := A + B_1(B_1^T X_k) - B_2(B_2^T X_k)$.
   - (ii) Solve the ARE
   $$\mathcal{R}(X_k) + A_k^T Z_k + Z_k A_k - Z_k B_2 B_2^T Z_k = 0.$$
   - (iii) Set $X_{k+1} := X_k + Z_k$.
   - (iv) IF $\|B_1^T Z_k\|_2 <$ tol THEN Stop.

Remark. ARE for $k = 0$ is the standard LQR/H_2 ARE.

Riccati Iterations

[Lanzon/Feng/B.D.O. Anderson 2007 (Proc. ECC 2007)]

Theorem [Lanzon/Feng/B.D.O. Anderson 2007]

If

- (A, B_2) stabilizable,
- (A, C) has no unobservable purely imaginary modes, and
- $\exists$ stabilizing solution X_-,

then

a) $(A + B_1 B_1^T X_k, B_2)$ stabilizable for all $k = 0, 1, \ldots,$

b) $Z_k \geq 0$ for all $k = 0, 1, \ldots,$

c) $A + B_1 B_1^T X_k - B_2 B_2^T X_{k+1}$ is Hurwitz for all $k = 0, 1, \ldots,$

d) $\mathcal{R}(X_{k+1}) = Z_k B_1 B_1^T Z_k$ for all $k = 0, 1, \ldots,$

e) $X_- \geq \ldots \geq X_{k+1} \geq X_k \geq \ldots \geq 0.$

f) If $\exists \lim_{k\to\infty} X_k =: \underline{X}$, then $\underline{X} = X_-$, and

g) convergence is locally quadratic.

Riccati Iterations

[Lanzon/Feng/B.D.O. Anderson 2007 (Proc. ECC 2007)]

Riccati iteration – low-rank version [B. 2008]

1. Solve the ARE
$$C^TC + A^TZ_0 + Z_0A - Z_0B_2B_2^TZ_0 = 0$$
using Newton-ADI, yielding Y_0 with $Z_0 \approx Y_0Y_0^T$.
2. Set $R_1 := Y_0$. $\{\%\ R_1R_1^T \approx X_1.\}$
3. FOR $k = 1, 2, \ldots,$
   - (i) Set $A_k := A + B_1(B_1^TR_k)R_k^T - B_2(B_2^TR_k)R_k^T$.
   - (ii) Solve the ARE
   $$Y_{k-1}(Y_{k-1}^TB_1)(B_1^TY_{k-1})Y_{k-1}^T + A_k^TZ_k + Z_kA_k - Z_kB_2B_2^TZ_k = 0$$
   using Newton-ADI, yielding Y_k with $Z_k \approx Y_kY_k^T$.
   - (iii) Set $R_{k+1} := \texttt{rrqr}([R_k,\ Y_k], \tau)$. $\{\%\ R_{k+1}R_{k+1}^T \approx X_{k+1}\}$
   - (iv) IF $\|(B_1^TY_k)Y_k^T\|_2 <$ tol THEN Stop.

AREs with Indefinite Hessian

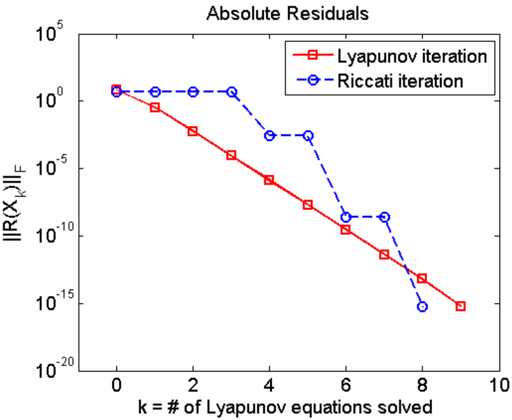
Numerical example

- Trivial example ($n = 2$) from [Cherfi/Abou-Kandil/Bourles 2005].
- Compare convergence of Lyapunov and Riccati iterations.
- Solution of standard AREs with Newton's method.

AREs with Indefinite Hessian

Numerical example

- Trivial example ($n = 2$) from [CHERFI/ABOU-KANDIL/BOURLES 2005].
- Compare convergence of Lyapunov and Riccati iterations.
- Solution of standard AREs with Newton's method.



Software

Lyapack [Penzl 2000]

MATLAB toolbox for solving

- Lyapunov equations and algebraic Riccati equations,
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Main work horse: Low-rank ADI and Newton-ADI iterations.

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MESS – Matrix Equations Sparse Solvers [Saak/Mena/B. 2008]

- Extended and revised version of Lyapack.
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Conclusions and Open Problems

- Low-rank Newton-ADI is a powerful and reliable method for solving large-scale AREs with semidefinite Hessian.
- Software is available in MATLAB toolbox Lyapack and its successor MESS.
- LQR problems on finite-time horizons can be solved using large-scale DRE solvers in MESS.
- Low-rank Riccati iteration yields a reliable and efficient method for large-scale AREs with indefinite Hessian, useful, e.g., for H_∞ optimization of PDE control problems.
- Low-rank Lyapunov iteration is an extremely simple variant for large-scale problems, but exhibits slower convergence and requires difficult-to-compute initial value.

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- To-Do list:
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  - computation of stabilizing initial guess.
    (If hierarchical grid structure is available, a multigrid approach is possible, other approaches based on "cheaper" matrix equations under development.)
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  ...for AREs with indefinite Hessian:

  - Implement Riccati iteration in Lyapack/MESS style.
  - More numerical tests.
  - Re-write Riccati iteration as feedback iteration.
  - Efficient computation of initial value for Lyapunov iterations?
  - $\exists$ perturbed Hessian so that Lyapunov iteration quadratically convergent?

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