Model order reduction of mechanical systems subjected to moving loads by the approximation of the input

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Elastic multibody systems (EMBS) → EMBS with moving loads

- **Vehicle-bridge interaction**

- **Working gears**

- **Cableways**

etc.
PDEs $\xrightarrow{FEM}$ ODEs

Example

\[ a_0 \frac{\partial^4}{\partial x^4} w(x,t) + a_1 \frac{\partial^2}{\partial t^2} w(x,t) + a_2 \frac{\partial}{\partial t} w(x,t) = \rho(x,t)u(t) \]

\[ b(t) = \begin{bmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_N(t) \end{bmatrix}, \quad b_i(t) = \int_0^l \rho(x,t)\phi_i(x)dx, \quad i = 1, \ldots, N \]

\[ M\ddot{q}(t) + D\dot{q}(t) + Kq(t) = b(t)u(t) \]

\[ y(t) = C(t)q(t) \]

\[ M, D, K \in \mathbb{R}^{N \times N}, \quad q \in \mathbb{R}^N, \quad b(t) \in \mathbb{R}^N, \quad C(t) \in \mathbb{R}^{p \times N}, \quad y(t) \in \mathbb{R}^p \]

many forces $\xrightarrow{I.}$ input matrix $B(t) \in \mathbb{R}^{N \times m}$ instead of $b(t)$
High computational cost

Model order reduction (MOR)

**MOR by projection:** \( q(t) \approx V \tilde{q}(t) \), \( \tilde{q} \in \mathbb{R}^r \), \( r \ll N \)

\[
W^T M V \ddot{q}(t) + W^T D V \dot{q}(t) + W^T K V \tilde{q}(t) = W^T B(t) u(t)
\]

\[
\tilde{M} \in \mathbb{R}^{r \times r}, \quad \tilde{D} \in \mathbb{R}^{r \times r}, \quad \tilde{K} \in \mathbb{R}^{r \times r}, \quad \tilde{B}(t) \in \mathbb{R}^{r \times m}
\]

\[
\tilde{y}(t) = C(t) V \tilde{q}(t)
\]

\[
\tilde{y}(t) \in \mathbb{R}^p, \quad \tilde{C}(t) \in \mathbb{R}^{p \times r}
\]

Systems with time-varying input and/or output matrices:

\[
V, W - ???
\]
II. Approximation of the input matrix

\[ \rho(x,t) = g(x - \zeta(t)), \quad \zeta(t) \text{ - a position of a «centre of force» at } t \]

\[ \zeta(t) \in \Omega \subseteq [0, l] \]

\[ b(t) = \begin{bmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_N(t) \end{bmatrix}, \quad b_i(t) = \int_0^l g(x - \zeta(t)) \phi_i(x) \, dx, \quad i=1,\ldots,N \]

Consider a SISO system

\[ M \ddot{q}(t) + D \dot{q}(t) + K q(t) = b(t) u(t) \]

\[ y(t) = b^T(t) q(t) \quad \text{with} \quad t \in [0, T] \]

Naive approach:

\[ M \ddot{q}(t) + D \dot{q}(t) + K q(t) = I(b(t) u(t)) \]

\[ y(t) = b^T(t) q(t) \]

**Difficulty:** many inputs
II. Approximation of the input matrix

**Goal:** approximate $b(t)$ in a lower dimension subspace

\[ b(t) \approx \hat{B} \chi(\zeta(t)) = \sum_{i=1}^{n} \hat{b}_i \chi_i(\zeta(t)), \quad n \ll N \]

\[ M \ddot{q}(t) + D \dot{q}(t) + K q(t) = \hat{B} \dot{u}(t) \quad \text{with} \quad \dot{u}(t) = \chi(\zeta(t)) u(t) \]
\[ \ddot{y}(t) = \hat{B}^{T} \dot{q}(t) \]

**Note:** $y(t) \approx \chi(\zeta(t))^{T} \ddot{y}(t)$

**Error bound:**

\[ \left\| \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix} - \begin{bmatrix} \dot{q}(t) \\ \ddot{q}(t) \end{bmatrix} \right\|_\infty \leq \eta \left\| b - \hat{B} \chi \right\|_\infty \]

**Two approximation approaches:**

1. given the matrix $\hat{B}$, find the vector $\chi(\zeta)$
2. given the vector $\chi(\zeta)$, find the matrix $\hat{B}$

such that $\left\| b - \hat{B} \chi \right\|_\infty \rightarrow \min$
II. Approximation of the input matrix

\[ b(t) = \begin{bmatrix} \varphi_1(\xi(t)) \\ \varphi_2(\xi(t)) \\ \vdots \\ \varphi_N(\xi(t)) \end{bmatrix} \approx \begin{bmatrix} \hat{b}_{11} & \cdots & \hat{b}_{1n} \\ \hat{b}_{21} & \cdots & \hat{b}_{2n} \\ \vdots & \ddots & \vdots \\ \hat{b}_{N1} & \cdots & \hat{b}_{Nn} \end{bmatrix} \begin{bmatrix} \chi_1(\xi(t)) \\ \chi_2(\xi(t)) \\ \vdots \\ \chi_N(\xi(t)) \end{bmatrix} = \hat{B} \chi(\xi(t)) \]

- **approximation by polynomial expansion** \( \varphi_i(x) \approx \sum_{j=1}^{n} \hat{b}_{ij} P_{j-1}(x), \quad i=1,\ldots,N \),
  where \( P_0(x),\ldots,P_{n-1}(x) \) are orthogonal polynomials;

- **B-spline interpolation** \( \varphi_i(x) \approx \sum_{j=1}^{n} \hat{b}_{ij} \beta_{j-2}(x), \quad i=1,\ldots,N \)
  where \( \beta_{-1}(x),\ldots,\beta_{n-2}(x) \) are B-splines;

- **linear least square method (LLSM)** \( \varphi_i(x) = \varphi_i^{(N)}(x) \approx \sum_{j=1}^{n} \hat{b}_{ij} \phi_j^{(n)}(x), \quad i=1,\ldots,N \)
  where \( \phi_1^{(n)}(x),\ldots,\phi_n^{(n)}(x) \) are FEM basis functions on a coarse grid;

- **empirical interpolation method (EIM)**
  [Barrault, Maday, Nguyen, Patera, 2004]
III. Model order reduction of mechanical systems

Balanced truncation
- solving Lyapunov equations is required
- use SO-LR-ADI method specially adapted for second-order systems [Benner, Kürschner, Saak, 2012]

But, for mechanical systems with a weak damping, this method converges very slowly

Krylov subspace methods
- SOAR [Bai, Su, 2005; Salimbahrami, 2005]
- SOR-IRKA, SO-IRKA [Wyatt, 2012]
- AORA [Lee, Chu, Feng, 2004; Bodendieck, Bollhöfer, 2013]
- MIRKA [Soppa, 2011]

Choice of interpolation points and directions for second-order systems is still unclear

Our approach: subspace acceleration poles finding combined with an extension of a frequency range
Test model with a moving load: 1D Euler-Bernoulli beam equation

\[
\rho A \frac{\partial^2}{\partial t^2} w(x,t) + 2 \rho A \omega_d \frac{\partial}{\partial t} w(x,t) + EI \frac{\partial^4}{\partial x^4} w(x,t) = \delta(x-\zeta(t)) u
\]

\((x,t) \in (0,l) \times (0,T) \) (has an analytical solution)  \[\text{[Fryba, 1999]}\]

- \(w(x,t)\) is a vertical deflection of the beam
- \(\delta(x-\zeta(t))\) is a point force density
- \(\nu\) is a velocity of the moving load
- \(\zeta(t)=\nu t\) is an instantaneous position of a force
- \(u\) is a magnitude of the moving load
- \(\rho\) is a mass density
- \(A\) is a cross section area
- \(\omega_d\) is a circular frequency of damping
- \(E\) is a Young modulus
- \(I\) is an area moment of inertia

with simply supported ends of the beam

\[
w(0,t)=0, \quad \frac{\partial^2}{\partial x^2} w(0,t)=0,
\]

\[
w(l,t)=0, \quad \frac{\partial^2}{\partial x^2} w(l,t)=0
\]

and initial conditions

\[
w(x,0)=0, \quad \frac{\partial}{\partial t} w(x,0)=0
\]
input distribution vector

\[
b(t) = \begin{bmatrix}
  b_1(t) \\
  b_2(t) \\
  \vdots \\
  b_N(t)
\end{bmatrix}, \quad b_i(t) = \int_0^l \delta(x - \zeta(t)) \phi_i(x) \, dx = \phi_i(\zeta(t)), \quad i = 1, \ldots, N,
\]

where \( \phi_i(x) \) is a finite element method basis function corresponded to some node and \( \zeta(t) \in \Omega = [0, l] \)

\[
M \ddot{q}(t) + D \dot{q}(t) + K q(t) = b(t)u
\]

\[
y(t) = b^T(t)q(t)
\]

\( b(t) \) - moving input distribution \[ b(t)u \) - moving load

\( c(t) = b^T(t) \) - moving output distribution \[ y(t) \) - moving observation
Approximation of FEM basis functions

\[ \phi_i(\zeta(t)) \approx \sum_{j=1}^{n} \hat{b}_{ij} \chi_j(t), \quad j = 1, \ldots, N, \quad n \ll N \]
IV. Numerical experiments

Approximated output by approximations of the input

\[ N = 5000, \quad n = 50 \]

\[ \phi_i(\xi(t)) \approx \sum_{j=1}^{50} \hat{b}_{ij} \chi_j(t), \quad i=1, \ldots, 5000 \]
Approximated output by approximations of the input

\[ N = 50, \quad n = 50 \]
**Approximated output of the reduced system with moving load**

\[ N = 5000, \quad n = 50, \quad r = 20 \]

*Reduction is carried out by the software MatMorembs*

http://www.itm.uni-stuttgart.de/research/model_reduction/MOREMBS_MatMorembs_en.php

Eberhard, Lehner, Fehr, Nowakowski, Fischer, Kürschner et al.
V. Conclusion

It was considered:

- second-order systems with moving load
- approximation of time-varying input matrix
- model reduction methods for mechanical systems

Further work:

- search of optimal methods to reduction of mechanical systems
- search of new approaches to model reduction of systems with moving load
Further work:

- consideration of more realistic models

for example, a coupled bridge-vehicle system

beam subjected to a moving two-axle system

Thank you for your attention!