Non-Conforming Finite Elements and Riccati-Based Feedback Stabilization of the Stokes Equations

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Goal

\[ \text{CC + A'X +XA} - \text{XB}^2 = 0 \]

K = -BX

Problem Setting

Motivation:
- Stabilization of flows described by Navier-Stokes equations (NSE)
- \[ \begin{align*}
&\frac{\partial v}{\partial t} - \nu \Delta v + v \cdot \nabla v + \nabla p = f, \\
&\nabla \cdot v = 0
\end{align*} \]

(1)

to steady-state solution, with \( \Omega \subset \mathbb{R}^d \), \( d = 2, 3 \), the velocity field \( v(x, t) \) \( \in \mathbb{R}^d \), the pressure \( p(x, t) \) \( \in \mathbb{R} \), the time \( t \in (0, \infty) \), the spatial variable \( x \in \Omega \), and the Reynolds number \( Re \) \( \in \mathbb{R}^d \).
- Construction based on associated linear quadratic control problem (LQR) for boundary control [4]
- Numerical treatment for 2D case with linearized NSE described in [1].

Here: Stokes equations

\[ \begin{align*}
&\frac{\partial v}{\partial t} - \nu \Delta v + \nabla p = f, \\
&\nabla \cdot v = 0
\end{align*} \]

(2)

Following [3] equation (4) is the semi discretized formulation of (2) including boundary data and projected to the manifold of divergence free discrete functions.

The pair \((\mathcal{E}, \mathcal{T})\) then implements the semi discretized, projected spatial differential operator from (2).

For \( i = 1 \) and \( X_0 = 0 \) for every column in \( V_i \) equation (6), for (5) respectively corresponds to solving a modified stationary Stokes problem:

\[ \begin{align*}
&\frac{1}{Re} \left( \nabla (v \cdot v) + \nu \nabla^2 v + \nu (\nabla v \cdot v + v \nabla v) \right) \\
&\nabla \cdot v = 0
\end{align*} \]

(7)

for test functions \( \varphi \in H^1(\Omega) \) – respecting the boundary conditions – and \( v \in \mathcal{E}(\mathcal{T}) \), in the evaluation of the i-th column of (6)/(5).

Similarly applications of \( \mathcal{A} \) and \( \mathcal{M} \) can be pulled back to the weak formulation level.

Advantages:
- (7) allows higher flexibility of formulation (e.g., adapting [3])
- possibility to work matrix free
- parallel implementations can exploit full FEM, PDE or domain features.

Abstract Problem Setting:
- Stabilization of flows described by Navier-Stokes equations (NSE)
- Finite element discretization of (2) yields

\[ \begin{align*}
&\mathcal{M} = A + G \cdot p + B \cdot u, \\
&0 = C' \cdot x
\end{align*} \]

(3)

with
- discretized velocity \( \mathbf{u}(t) \in \mathbb{R}^6 \) and pressure \( p(t) \in \mathbb{R}^6 \)
- symmetric positive definite mass matrix \( M \in \mathbb{R}^{6 \times 6} \)
- system matrix \( \Lambda \in \mathbb{R}^{6 \times 6} \) (symmetric for Stokes) and
- discretized gradient \( G \in \mathbb{R}^{6 \times 6} \) of rank \( r_p \).

In the context of an LQR problem one additionally gets
- the input matrix \( B \in \mathbb{R}^{6 \times 6} \)
- the output \( u(t) \in \mathbb{R}^6 \)

which describe the boundary control. Partial observation furthermore leads to
the output \( y(t) \in \mathbb{R}^6 \) and
the output matrix \( C \in \mathbb{R}^{6 \times 6} \).

Semi Discretized Problem Setting:
- Newton Kleinman Method
- Approximate X solving:

\[ \begin{align*}
&\mathcal{C} \cdot C + A'X +XA - \mathcal{M}X = X \mathcal{M} + X \mathcal{M}X
\end{align*} \]

in step / solve the Lyapunov equation:

\[ \begin{align*}
&(A' - X)X + \mathcal{M}X + \mathcal{M}X(A - BK_{\mu}) = -C_i C_i^T
\end{align*} \]

where \( K_{\mu} = B^T X \mathcal{M} G + C_i G_i^T \).

Applying the low rank ADI algorithm requires to solve

\[ \begin{align*}
&\mathcal{A} + \mu \mathcal{M} \mathcal{V} = \mathcal{M} \mathcal{V} \mathcal{L}
\end{align*} \]

(6)

with \( \mathcal{A} = A' - X \) \( \mathcal{L} \)
for a possibly complex \( \mu \) in each step.

Solve (5) instead of (6) to increase efficiency. Requires:
- Sherman-Morrison-Woodbury formula,
- block preconditioning (e.g., [2]),
- investigation of required accuracies, i.e., inexact Newton-Kleinman-ADI.

New here:
- Investigation of special finite elements that help ensuring “divergence free” condition for inexact solvers
- Interpretation of (5) in terms of the original PDE system.

Implicit Index Reduction
To rewrite the DAE system (3) with differential index two as a generalized state space system, we use the projector

\[ \begin{align*}
&\tilde{\mathcal{M}} = A \mathcal{A}^T + \mathcal{M}^T \mathcal{A} \mathcal{A} \mathcal{M} + \mathcal{M} \mathcal{A}^T \mathcal{A} \mathcal{M} \mathcal{A} \mathcal{M}^T
\end{align*} \]

defined in [3]. The projected ODE system is of the form

\[ \begin{align*}
&M\dot{x} = \tilde{\mathcal{M}} x + \mathcal{M} y,
\end{align*} \]

(4)

with \( M = \mathcal{M}^T > 0 \) and \( \mathcal{M} \in \mathbb{R}^{6 \times 6} \).

To solve the algebraic Riccati equation associated to the system (4) we use a Newton-ADI-method. Instead of solving the projected dense Lyapunov equations in the innermost loop, we use [3, Lemma 5.2] and have to solve the saddle point system

\[ \begin{align*}
&A^T + \mu \mathcal{M}^T G \mathcal{C}_1 \mathcal{C}_1^T = 0
\end{align*} \]

(5)

for a couple of right hand sides \( \mathcal{Y} \) and a different shift \( \mu \) in each ADI step during each Newton step.

Contribution Details

Numerical Results

Evolution of \( \mathcal{J} \) control and output \( \Rightarrow \) for the cost function

\[ \mathcal{J}(x,u) = \frac{1}{2} \int_0 T ||x||^2 + \frac{1}{2} ||u||^2 dt. \]

References


